Reducing concept lattices by means of a weaker notion of congruence

Roberto G. Aragón, Jesús Medina, Eloísa Ramírez-Poussa

Department of Mathematics, University of Cádiz. Spain
Email: {roberto.aragon,jesus.medina,eloisa.ramirez}@uca.es

Abstract

Attribute and size reductions are key issues in formal concept analysis. In this paper, we consider a special kind of equivalence relation to reduce concept lattices, which will be called local congruence. This equivalence relation is based on the notion of congruence on lattices, with the goal of losing as less information as possible and being suitable for the reduction of concept lattices. We analyze how the equivalence classes obtained from a local congruence can be ordered. Moreover, different properties related to the algebraic structure of the whole set of local congruences are also presented. Finally, a procedure to reduce concept lattices by the new weaker notion of congruence is introduced. This procedure can be applied to the classical and fuzzy formal concept analysis frameworks.

Keywords: Formal concept analysis, size concept lattice reduction, attribute reduction, congruence relation, fuzzy sets

1. Introduction

Formal Concept Analysis (FCA) is an exploratory data analysis technique, introduced by Ganter and Wille in [16], which has been widely studied from theoretical and applied perspectives. One of the key problems of formal concept analysis is to reduce the computational complexity of computing the complete lattice associated with the considered formal context (dataset). One procedure to address this problem is to find mechanisms to reduce the number of attributes, preserving the most important information contained in the context. Indeed, we can find many works which analyze different mechanisms that chase this goal [1, 3, 11, 12, 13, 14, 19, 22, 24, 25].

Recently, in [8, 7], the authors have presented novel mechanisms to reduce classical and fuzzy formal contexts based on the reduction philosophy considered in Rough Set Theory, which is another mathematical theory closely related to FCA [21, 22]. In the aforementioned papers, the authors exposed that when the number of attributes of a context is reduced, an equivalence relation on the set of concepts of the original concept lattice is induced, both in the classical and fuzzy cases. In addition, they also showed that the resulting equivalence classes have the structure of join-semilattices with a maximum element. In the light of the results presented in [8, 7], it is natural to ask how we could complement the introduced reduction mechanisms in order to ensure that the obtained equivalence classes are closed algebraic structures.

Specifically, we are interested in obtaining equivalence classes satisfying that they are convex sublattices of the original concept lattice. This target can be reached by considering the notion of congruence relation on lattices [9, 15, 17, 18]. Although congruence relations within the environment of FCA have not been studied extensively, we can find some works that analyze the use of congruence relations within this mathematical theory. For example, congruence relations have been applied in lattice/context decomposition as Atlas decomposition [16], the subdirect decomposition [27] or the reverse doubling construction [26]. In addition, the links between implications and congruence relations have been analyzed in [28] and congruence relation have proved to be suitable to handle with inconsistent formal decision contexts [20].

However, a significant amount of information can be lost when congruence relations are considered to reduce concept lattices, due mainly to the restrictions imposed by the quadrilateral-closed property. In order to address this issue, in this paper we continue with the study presented in [5], in which we introduced a weaker notion of congruence by means of the elimination of the aforementioned restrictive property. We will analyze how the equivalence classes obtained from a local congruence can be ordered and so, if some hierarchy exists among the clusters provided by the local congruence. Then, since different local congruence relations can be defined on a concept lattice, we will go further to a meta level, studying the algebraic structure of the set of all local congruences that can be defined on a lattice and other interesting properties. Finally, based on the obtained results, a procedure to reduce concept lattices is presented by using local congruence relations. One of the advantages provided by this procedure is that it can be applied both in the classical and fuzzy generalizations of formal concept analysis. In this work, examples to illustrate the proposed procedure are also included. The introduced examples consider classical FCA, as well as the fuzzy generalization of this theory provided by the multi-adjoint framework [23]. These examples also confirm that the use of this kind of equivalence relations is more suitable for this task than the use of congruences, since the amount of lost information is minimized.

The paper is organized as follows: Section 2 reviews some preliminary notions related to lattice theory, congruences on lattices and formal concept analysis, which are necessary to follow this work. In Section 3 the notion of local congruence and several properties are introduced. A study on the ordering among the equivalence classes obtained from local congruences is included in Section 4. The properties related to the algebraic structure of the whole set of local congruences that can be defined on a lattice and to principal local congruences are given in Section 5. Section 6 presents a mechanism to reduce concept lattices based on the use of local congruences. The paper ends in Section 7, showing some conclusions and proposing diverse future challenges.

2. Preliminaries

In this section, some preliminary notions used in this work will be recalled and we will state the considered notation.

We will consider a lattice as an algebraic structure $(L, \wedge, \vee)$ and as an ordered set $(L, \preceq)$. It is well known that these two points of view are equivalent, since we can define the two operators infimum and supremum from the partial order and vice versa, see The Connecting Lemma in [15]. Therefore, we will write $(L, \wedge, \vee)$ or $(L, \preceq)$ indistinctly, depending on the most suitable point of view in each case.

In this work, we are interested in defining equivalence relations on complete lattices. We will write $(a, b) \in R$ with $a, b \in A$ to indicate that a and b are related under the binary relation R. Notice that an equivalence relation R on A gives rise to a partition of A, whose subsets are the equivalence classes of R. The set of all the equivalence classes of R is called *quotient set* and it is denoted as A/R. Equivalently, a partition of A gives rise to an equivalence relation whose equivalence classes are the subsets of the partition.

From this point forward if $\rho \subseteq A \times A$ is an equivalence relation on a set A, we will denote the equivalence class of an element $a \in A$ as $[a]_\rho = \{b \in$

$A \mid (a, b) \in \rho\}$.

2.1. Congruence on lattices

This section introduces the notion of congruence on a lattice and some features which are essential to develop our work. First of all, we present the definition of equivalence relation that is compatible with the operation of the algebraic structure.

Definition 1. We say that an equivalence relation θ on a given lattice $(L, \wedge, \vee)$ is *compatible* with the supremum $\vee$ and the infimum $\wedge$ of the lattice if, for all $a, b, c, d \in L$,

$$(a, b) \in \theta \quad \text{and} \quad (c, d) \in \theta$$

imply

$$(a \vee c, b \vee d) \in \theta \quad \text{and} \quad (a \wedge c, b \wedge d) \in \theta$$

We can now state the definition of congruence on a lattice.

Definition 2. Given a lattice $(L, \wedge, \vee)$, we say that an equivalence relation on L, which is compatible with both the supremum and the infimum of $(L, \wedge, \vee)$ is a *congruence* on L.

Now, we introduce the notion of quotient lattice from a congruence based on the operations of the original lattice.

Definition 3. Given an equivalence relation θ on a lattice $(L, \wedge, \vee)$, two operators $\vee_\theta$ and $\wedge_\theta$ on the set of equivalence classes $L/\theta = \{[a]_\theta \mid a \in L\}$, for all $a, b \in L$, are defined as follows

$$[a]_\theta \vee_\theta [b]_\theta = [a \vee b]_\theta \text{ and } [a]_\theta \wedge_\theta [b]_\theta = [a \wedge b]_\theta.$$

$\vee_\theta$ and $\wedge_\theta$ are well defined on L/θ if and only if θ is a congruence.

When θ is a congruence on L, we call $\langle L/\theta, \vee_\theta, \wedge_\theta \rangle$ *the quotient lattice of L modulo* θ.

The following lemma is useful when calculating with congruences.

Lemma 4 ([15]). *Given a lattice* $(L, \wedge, \vee)$ *we have that*

(i) *An equivalence relation θ on L is a congruence if and only if, for all $a, b, c \in L$,*

$$(a, b) \in \theta \text{ implies } (a \vee c, b \vee c) \in \theta \quad \text{and} \quad (a \wedge c, b \wedge c) \in \theta.$$

(ii) *Let θ be a congruence on L and $a, b, c \in L$.*
   (a) *If $(a, b) \in \theta$ and $a \preceq c \preceq b$, then $(a, c) \in \theta$.*
   (b) *$(a, b) \in \theta$ if and only if $(a \wedge b, a \vee b) \in \theta$.*

The equivalence classes of a congruence are convex sublattices of the original lattice and besides are quadrilateral-closed. Let us recall the meaning of notion of quadrilateral-closed. Let $(L, \preceq)$ be a lattice, an equivalence relation θ on $(L, \preceq)$ and suppose that $\{a, b, c, d\}$ is a subset of L composed of four elements forming a quadrilateral, then a, b and c, d are said to be *opposite sides of the quadrilateral* $\langle a, b; c, d\rangle$ (see Figure 1) if $a \prec b$, $c \prec d$ and either:

$$(a \vee d = b \text{ and } a \wedge d = c) \quad \text{or} \quad (b \vee c = d \text{ and } b \wedge c = a).$$

Therefore, *quadrilateral-closed* means that whenever given two opposite sides of a quadrilateral a, b and c, d, satisfying that a, b belong to an equivalence class, then c, d belong to another or the same equivalence class, that is, if $a, b \in [x]_\theta$, with $x \in L$ then there exists $y \in L$ such that $c, d \in [y]_\theta$.

Figure 1: Opposite sides of a quadrilateral.

The following result introduces an interesting characterization of congruence which will be fundamental for the purpose of this paper.

Theorem 5 ([15]). *Let $(L, \wedge, \vee)$ be a lattice and let θ be an equivalence relation on L. Then θ is a congruence if and only if*

(i) *each equivalence class of θ is a sublattice of L,*
(ii) *each equivalence class of θ is convex,*
(iii) *the equivalence classes of θ are quadrilateral-closed.*

The set of congruences on a lattice L, denoted as Con L, is a topped $\cap$-structure on $L \times L$. Hence Con L, ordered by inclusion, is a complete lattice. The least element and the greatest element are given by $\theta_\bot = \{(a,a) \mid a \in L\}$ and $\theta_\top = \{(a,b) \mid a,b \in L\}$, respectively.

Given a lattice $(L, \wedge, \vee)$ and two elements $a, b \in L$, the least congruence satisfying that a and b are related is denoted as $\theta_{(a,b)}$ and it is called the *principal congruence generated by* (a,b) and it is defined as follows

$$\theta_{(a,b)} = \bigwedge \{\theta \in \text{Con } L \mid (a,b) \in \theta\}.$$

Next lemma shows the importance of this definition.

Lemma 6 ([15]). *Let* $(L, \wedge, \vee)$ *be a lattice and* $\theta \in$ *Con* L. *Then*

$$\theta = \bigvee \{\theta_{(a,b)} \mid (a,b) \in \theta\}.$$

Therefore, principal congruences factorize any congruence.

2.2. Formal concept analysis

Since equivalence relations will be considered in this work to reduce concept lattices, basic definitions of FCA are recalled in order to understand the motivation and results presented in this paper.

Definition 7. A *context* is a triple (A, B, R) with a set of attributes A, a set of objects B and a crisp relationship $R \subseteq A \times B$. We will write $R(a,b) = 1$ when $(a,b) \in R$ and $R(a,b) = 0$ when $(a,b) \notin R$.

Furthermore, if we consider a context, two mappings, ${}^\uparrow \colon 2^B \to 2^A$ and ${}^\downarrow \colon 2^A \to 2^B$, can be defined for each $X \subseteq B$ and $Y \subseteq A$ as:

$$X^\uparrow = \{a \in A \mid (a,x) \in R, \text{for all } x \in X\} \quad (1)$$
$$Y^\downarrow = \{x \in B \mid (a,x) \in R, \text{for all } a \in Y\} \quad (2)$$

These operators form a Galois connection [15], which leads us to the following definition.

Definition 8. Given a context (A, B, R) and the operators $^\uparrow$ and $^\downarrow$ defined above. If for a pair (X,Y) with $X \subseteq B$ and $Y \subseteq A$, the equalities $X^\uparrow = Y$ and $Y^\downarrow = X$ hold, then the pair (X,Y) is called *concept*.

Given a pair of concepts (X_1, Y_1) and (X_2, Y_2), we say that $(X_1, Y_1) \leq (X_2, Y_2)$ if $X_1 \subseteq X_2$ ($Y_2 \subseteq Y_1$, equivalently). The set of all concepts with this ordering relation has the structure of a complete lattice, it is called *formal concept lattice* and it is denoted as $\mathcal{C}(A, B, R)$ [15, 16].

Now, we recall two results about reduction in FCA [8]. The first one shows that when we reduce the set of attribute of a formal context, an equivalence relation on the set of concepts of the original concept lattice is induced.

Proposition 9 ([8]). *Given a context (A, B, R) and a subset $D \subseteq A$. The set $\rho_D = \{((X_1, Y_1), (X_2, Y_2)) \mid (X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R), X_1^{\uparrow_D \downarrow} = X_2^{\uparrow_D \downarrow}\}$ is an equivalence relation. Where $^{\uparrow_D}$ denotes the concept-forming operator, given in Expresion (1), restricted to the subset of attributes $D \subseteq A$.*

The next result shows that every class of the equivalence relation defined above has the structure of a join semilattice with maximum element.

Proposition 10 ([8]). *Given a context (A, B, R), a subset $D \subseteq A$ and a class $[(X, Y)]_D$ of the quotient set $\mathcal{C}(A, B, R)/\rho_D$. The class $[(X, Y)]_D$ is a join semilattice with maximum element $(X^{\uparrow_D \downarrow}, X^{\uparrow_D \downarrow \uparrow})$.*

Hence, we cannot ensure that the classes are sublattices of the original concept lattice, as it was shown in Example 3.10 of [8]. Therefore, it is interesting to study when these classes are sublattices and the properties of the obtained reduction. These results have been extended to the fuzzy FCA framework of multi-adjoint concept lattices in [7]. This framework was introduced by Medina, Ojeda-Aciego and Ruiz-Calviño in [23] with the main goal of presenting a general and flexible FCA framework based on the multi-adjoint philosophy. Multi-adjoint concept lattice generalizes different fuzzy extension of FCA [4, 6, 10] and has widely been studied in diverse papers [2, 13, 14]. See the basic notions in [23].

In the multi-adjoint concept lattice framework a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$ needs to be fixed on which a context (A, B, R, σ) is defined and a concept lattice $\mathcal{M}(A, B, R, \sigma)$ is obtained. On this framework, the authors in [7] also proved that a reduction of the set of attributes induces an equivalence relation on $\mathcal{M}(A, B, R, \sigma)$, in which the equivalence classes are join-subsemilattices. This result will be recalled next, where $\uparrow_D$ and $\downarrow^D$ are the concept-forming operators associated with the subcontext $\mathcal{M}(D, B, R_{|D \times B}, \sigma)$, with $D \subseteq A$.

Proposition 11 ([7]). *Let $D \subseteq A$ be a subset of attributes. The set $\rho_D = \{(\langle g_1, f_1\rangle, \langle g_2, f_2\rangle) \mid \langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in \mathcal{M}(A, B, R, \sigma), g_1^{\uparrow_D \downarrow^D} = g_2^{\uparrow_D \downarrow^D}\}$ is an equivalence relation and every class $[\langle g, f\rangle]_D$ of $\mathcal{M}(A, B, R, \sigma)/\rho_D$ is a join-semilattice with maximum element $\langle g^{\uparrow_D \downarrow^D}, g^{\uparrow_D \downarrow^D \uparrow}\rangle$.*

Once we have recalled the required preliminary notions, the main contributions of this work are presented in the following section.

3. Weakening the notion of congruence

In this section, we present a weaker notion of congruence relation in order to complement reduction mechanisms in FCA. As we have recalled, any reduction of the set of attributes generates a partition in the set of concepts associated with the original context, where the obtained equivalence classes may not form sublattices of the original concept lattice. Our interest lies in generating groups of concepts with a closed algebraic structure by complementing the reductions given in FCA.

This goal can be achieved through the notion of congruence relation on lattices. Therefore, we will consider the use of congruence relations to reduce concept lattices, and we will analyze the obtained results. In particular, we are interested in the least congruence whose equivalence classes contain the equivalence classes induced by an attribute reduction of a context. Usually, this reduction is given by reducts which are minimal subsets of attributes preserving the information in the dataset. More details are included in [8].

Next, we illustrate the result of applying congruences through a practical example considered in [8]. In this example, we will show that the equivalence classes induced by a reduction procedure may be noticeably different from the ones provided by the least congruence containing the partition induced by the reduction, as a consequence, this difference would entail a relevant loss of information.

Example 12. Given the formal context (A, B, R) displayed in Table 1, where the set of objects in B are the planets of the Solar System together with the dwarf planet Pluto, that is $B = \{$Mercury (M), Venus (V), Earth (E), Mars (Ma), Jupiter (J), Saturn (S), Uranus (U), Neptune (N), Pluto (P)$\}$ and the set of attributes $A = \{$small size (ss), medium size (ms), large size (ls), near sun (ns), far sun, (fs), moon yes (my), moon no (mn)$\}$.

R	M	V	E	Ma	J	S	U	N	P
small size	1	1	1	1	0	0	0	0	1
medium size	0	0	0	0	0	0	1	1	0
large size	0	0	0	0	1	1	0	0	0
near sun	1	1	1	1	0	0	0	0	0
far sun	0	0	0	0	1	1	1	1	1
moon yes	0	0	1	1	1	1	1	1	1
moon no	1	1	0	0	0	0	0	0	0

Table 1: Relation of Example 12.

In the left side of Figure 2, it is displayed the concept lattice from the given context. In [8], the rough set reduct $D_1 = \{\text{small size}, \text{medium size}, \text{near sun}, \text{moon yes}\}$ was considered to reduce the context. According to Proposition 9, this reduction makes that the concepts of the original concept lattice are grouped in equivalence classes which are represented in the middle of Figure 2 by means of a Venn diagram. We know that each equivalence class has the structure of a join semilattice with maximum element as it is stated in Proposition 10.

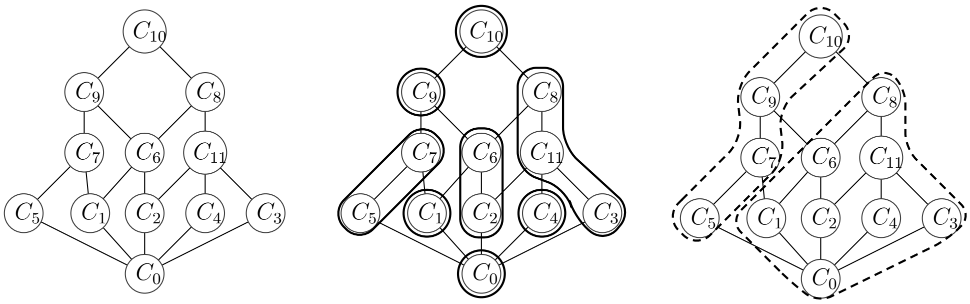

Figure 2: The original concept lattice (left), the obtained reduction in [8] (middle) and the least congruence containing the previously reduction (right).

We can bring together the reduction given in FCA and congruences, finding the least congruence such that each equivalence class induced by the reduction of the context is included in one equivalence class provided by the congruence relation. This least congruence is shown in the right side of Figure 2 by means of a dashed Venn diagram. As we can see in Figure 2, this congruence relation is composed of only two equivalence classes, since it has grouped too many concepts in each class. Consequently, in this case, the use of congruences entails a relevant loss of information, which is not convenient in any process of data analysis. □

The result obtained in the previous example reveals the necessity of a weaker notion of congruence removing the quadrilateral-closed property and preserving the other two properties in the characterization given in Theorem 5. This weaker notion is introduced in the following definition.

Definition 13. Given a lattice $(L, \preceq)$, we say that an equivalence relation δ on L is a *local congruence* if the following properties hold:

(i) each equivalence class of δ is a sublattice of L,

(ii) each equivalence class of δ is convex.

Remark 14. Clearly, the attribute reduction of the concept lattice introduced in Example 12 provides a local congruence (concept lattice in the middle of Figure 2). Therefore, the introduced notion offers a better reduction than the one provided using congruences, aggregating as less concepts (information) as possible. Moreover, since in this particular case the reduction already produces equivalence classes that are convex sublattices, then the amount of lost information with this weaker notion of congruence is minimized as much as possible.

Although this new definition is a weak definition of the notion of congruence, the name of weak-congruence has already been used in the literature [29, 30, 31, 32] in order to define congruences without the reflexivity property, that is, a weak congruence is a symmetry, transitivity and compatible relation. Therefore, another suitable name has been considered in this paper to the introduced general notion of congruence, whose justification will be introduced after the next direct characterization of Definition 13 in terms of the equivalence relation δ.

Proposition 15. *Given a lattice $(L, \preceq)$ and an equivalence relation δ on L, the relation δ is a local congruence on L if and only if, for each $a, b, c \in L$, the following properties hold:*

(i) If $(a, b) \in \delta$ and $a \preceq c \preceq b$, then $(a, c) \in \delta$.

(ii) $(a, b) \in \delta$ if and only if $(a \wedge b, a \vee b) \in \delta$.

Proof. The proof holds directly from the definition of local congruence. □

This result will be basic to introduce the following characterization, which generalizes Lemma 4(i) and motivates the considered notion.

Proposition 16. *Given a lattice* $(L, \preceq)$ *we have that an equivalence relation* δ *on* L *is a local congruence if and only if, for all* $a, b, c \in L$*, if* $(a, b) \in \delta$ *and* $a \wedge b \preceq c \preceq a \vee b$*, then we have that*

$$(a \vee c, b \vee c) \in \delta \quad \textit{and} \quad (a \wedge c, b \wedge c) \in \delta$$

Proof. Let us assume that δ is a local congruence on L and we consider $a, b, c \in L$ such that $(a, b) \in \delta$ and $a \wedge b \preceq c \preceq a \vee b$. Straightforwardly, by Proposition 15, we have that

$$(a, a \vee b) \in \delta \text{ and } (b, a \vee b) \in \delta. \tag{3}$$

In addition, since $a \wedge b \preceq c \preceq a \vee b$, by the supremum property, the following inequalities hold

$$a \vee (a \wedge b) \preceq a \vee c \preceq a \vee (a \vee b),$$

which is equivalent to

$$a \preceq a \vee c \preceq a \vee b.$$

Hence, as $(a, a \vee b) \in \delta$, by Proposition 15(i), we obtain that $(a, a \vee c) \in \delta$. Considering an analogous procedure to the previous one, we have that $(b, b \vee c) \in \delta$. Therefore, since $(a, a \vee c) \in \delta$ and $(b, b \vee c) \in \delta$, by the hypothesis $(a, b) \in \delta$ and the transitivity property of δ, we can assert that $(a \vee c, b \vee c) \in \delta$. Analogously, we have that $(a \wedge c, b \wedge c) \in \delta$.

Now, let us assume that δ is an equivalence relation on L, such that, if $(a, b) \in \delta$ and $a \wedge b \preceq c \preceq a \vee b$, then it satisfies that

$$(a \vee c, b \vee c) \in \delta \text{ and } (a \wedge c, b \wedge c) \in \delta$$

for all $a, b, c \in L$.

We will use Proposition 15 in order to prove that δ is a local congruence. If $(a, b) \in \delta$ and $a \preceq c \preceq b$ then, in particular, $a \wedge b \preceq a \preceq c \preceq b \preceq a \vee b$. Therefore, by hypothesis, we have that $(a \wedge c, b \wedge c) \in \delta$. Since $a \wedge c = a$ and $c \wedge b = c$, we obtain that $(a, c) \in \delta$. Hence, item (i) of Proposition 15 is satisfied.

In addition, for each $(a, b) \in \delta$ if we consider $c = a$, by hypothesis, we have that $(a \vee c, b \vee c) \in \delta$, that is, $(a, b \vee a) \in \delta$. Similarly, for $c = b$ we have that $(a \wedge b, b) \in \delta$. Hence, by the transitivity property of δ, we obtain that $(a \wedge b, a \vee b) \in \delta$. Following a similar reasoning, we can easily prove that, for all $a, b \in L$, if $(a \wedge b, a \vee b) \in \delta$ then $(a, b) \in \delta$. Therefore, item (ii) of Proposition 15 also holds.

Consequently, we can conclude that δ is a local congruence on L. □

Hence, the difference from the equivalence given in Lemma 4(i) is that in the aforementioned lemma, the element c is arbitrary in L, and in Proposition 16 is a *local* element bounded by $a \wedge b$ and $a \vee b$.

As we highlighted above, the particular algebraic structure of the equivalence classes is the most important property associated with the new notion. The set of these classes is formally defined next.

Definition 17. Let $(L, \preceq)$ be a lattice and δ a local congruence, the quotient set L/δ provides a partition of L, which is called *local congruence partition* (or *lc-partition* in short) of L and it is denoted as π_δ. The elements in the lc-partition π_δ are convex sublattices of L.

According to the previous definition, it can be noted that each equivalence class of the quotient set L/δ is a closed algebraic structure. Moreover, each local congruence relation univocally determines a local congruence partition and vice versa. As a consequence, both notions can be considered indistinctly.

In the following section, a formal definition of ordering among the classes of the quotient set of a local congruence will be studied.

4. The quotient set of a local congruence

Now, we focus on the equivalence classes of a quotient set provided by a local congruence. We are interested in studying how we can establish an ordering relation between these classes. The following definition will play a key role for this purpose.

Definition 18. Let $(L, \preceq)$ be a lattice and a local congruence δ on L.

(i) A sequence of elements of L, $(p_0, p_1, \ldots, p_n)$ with $n \geq 1$, is called a δ-*sequence*, denoted as $(p_0, p_n)_\delta$, if for each $i \in \{1, \ldots, n\}$ either $(p_{i-1}, p_i) \in \delta$ or $p_{i-1} \preceq p_i$ holds.

(ii) If a δ-sequence, $(p_0, p_n)_\delta$, satisfies that $p_0 = p_n$, then it is called a δ-*cycle*. In addition, if the δ-cycle satisfies that $[p_0]_\delta = [p_1]_\delta = \cdots = [p_n]_\delta$ is said to be *closed*.

The notions in Definition 18 are clarified in Figure 3 and Figure 4, where the triple vertical line that connects p_{i-1} with p_i means that they are related under the considered local congruence relation, that is, $(p_{i-1}, p_i) \in \delta$. The simple line indicates that the two elements are connected by means of the ordering relation defined on the lattice.

The following definition provides a first step to define a partial order on the quotient set provided by a local congruence.

Definition 19. Given a lattice $(L, \preceq)$ and a local congruence δ on L, we define a binary relation $\preceq_\delta$ on L/δ as follows:

$$[x]_\delta \preceq_\delta [y]_\delta \quad \text{if there exists a } \delta\text{-sequence } (x', y')_\delta$$

for some $x' \in [x]_\delta$ and $y' \in [y]_\delta$.

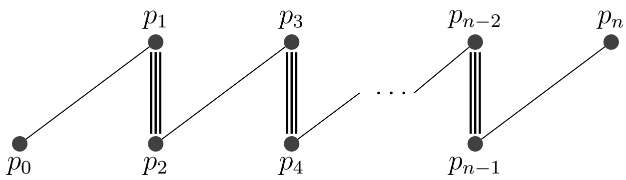

Figure 3: Example of δ-sequence.

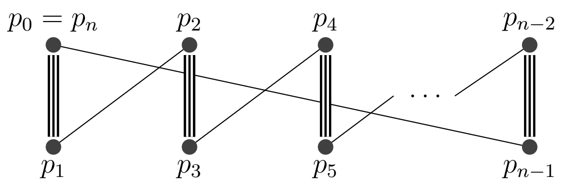

Figure 4: Example of δ-cycle.

Notice that the relation $\preceq_\delta$ given in Definition 19 is a preorder. Clearly, by definition, $\preceq_\delta$ is reflexive and transitive. However, the relation $\preceq_\delta$ is not a partial order since the antisymmetry property does not hold, for any local congruence in general. In the following example, we show a case in which the previously defined relation $\preceq_\delta$ does not satisfy the antisymmetry property.

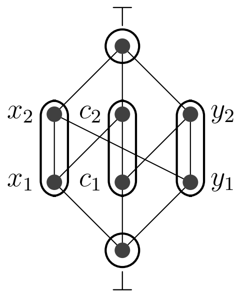

Figure 5: Example where $\preceq_\delta$ is not a partial order.

Example 20. Given the lattice $(L, \preceq)$ and the local congruence δ given in Figure 5. We have that the equivalence classes of δ are $[\top]_\delta = \{\top\}, [x_1]_\delta = \{x_1, x_2\}, [c_1]_\delta = \{c_1, c_2\}, [y_1]_\delta = \{y_1, y_2\}$ and $[\bot]_\delta = \{\bot\}$. It is easy to check that these equivalence classes are convex sublattices of L. Moreover, we can observe that $[x_1]_\delta \preceq_\delta [y_1]_\delta$ since there exists a δ-sequence, $(x_1, y_2)_\delta = (x_1, c_2, c_1, y_2)$, and also $[y_1]_\delta \preceq_\delta [x_1]_\delta$ since there also exists a δ-sequence, $(y_1, x_2)_\delta = (y_1, x_2)$, but $[x_1]_\delta \neq [y_1]_\delta$ and thus $\preceq_\delta$ is not antisymmetric. □

There are certain cases in which the relation $\preceq_\delta$ is a partial order, depending on the local congruence.

Example 21. Let $(L, \preceq)$ be a lattice isomorphic to the concept lattice given in Example 12 and δ the local congruence shown in the left side of Figure 6. In this case, the considered local congruence makes that the relation $\preceq_\delta$ satisfies the antisymmetry property and, consequently, the relation $\preceq_\delta$ is a partial order and the Hasse diagram of $(L/\delta, \preceq_\delta)$ can be given (right side of Figure 6). □

The following results state different conditions under which the relation $\preceq_\delta$ is a partial order.

Proposition 22. *Given a lattice $(L, \preceq)$ and a local congruence δ, if for any two equivalence classes $[x]_\delta, [y]_\delta \in L/\delta$ there exists only one class $[c]_\delta \in L/\delta$ such that $[x]_\delta \preceq_\delta [c]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [c]_\delta \preceq_\delta [x]_\delta$ satisfying that $x_1 \preceq c_1 \preceq y_1$ and $y_2 \preceq c_2 \preceq x_2$ with $x_1, x_2 \in [x]_\delta$, $c_1, c_2 \in [c]_\delta$ and $y_1, y_2 \in [y]_\delta$, then $[x]_\delta = [y]_\delta$.*

Proof. Let us consider two equivalence classes $[x]_\delta, [y]_\delta \in L/\delta$. Hence, there exists a class $[c]_\delta \in L/\delta$ such that $[x]_\delta \preceq_\delta [c]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta$

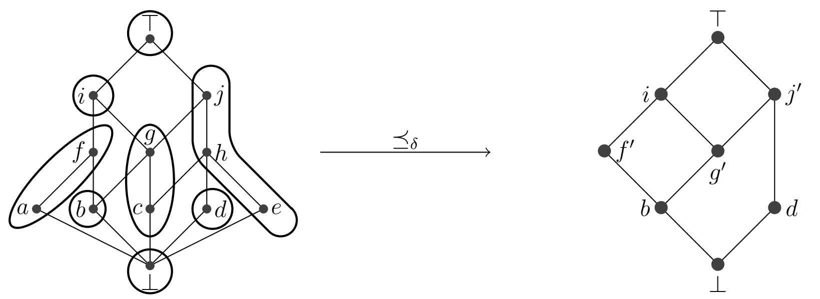

Figure 6: A local congruence δ on L (left) and its quotient set $(L/\delta, \preceq_\delta)$ (right).

$[c]_\delta \preceq_\delta [x]_\delta$ satisfying that $x_1 \preceq c_1 \preceq y_1$ and $y_2 \preceq c_2 \preceq x_2$ with $x_1, x_2 \in [x]_\delta$, $c_1, c_2 \in [c]_\delta$ and $y_1, y_2 \in [y]_\delta$. Then, we have that $[x]_\delta \preceq_\delta [c]_\delta$ and $[c]_\delta \preceq_\delta [x]_\delta$ and, since the classes of δ are sublattices of L, we also have that $x_1 \vee x_2$ and $c_1 \vee c_2$ exist and belong to the classes $[x]_\delta$ and $[c]_\delta$, respectively. In addition, we have that $x_1 \vee c_2 \preceq c_1 \vee c_2$ and $x_1 \vee c_2 \preceq x_1 \vee x_2$, thus $x_1 \vee c_2 \preceq (x_1 \vee x_2) \wedge (c_1 \vee c_2)$. Hence, $x_1 \preceq (x_1 \vee x_2) \wedge (c_1 \vee c_2) \preceq x_1 \vee x_2$ and $c_2 \preceq (x_1 \vee x_2) \wedge (c_1 \vee c_2) \preceq c_1 \vee c_2$, by the convexity of the classes we have that $(x_1 \vee x_2) \wedge (c_1 \vee c_2)$ belongs to both classes, which implies that both classes are just the same class: $[x]_\delta = [c]_\delta$.

We can proceed in an analogous way in order to prove that $[c]_\delta = [y]_\delta$. Hence, we obtain that $[x]_\delta = [c]_\delta = [y]_\delta$. □

From the previous proposition we obtain the following corollary.

Corollary 23. *Given a lattice $(L, \preceq)$ and a local congruence δ, if for any two equivalence classes $[x]_\delta, [y]_\delta \in L/\delta$ such that $[x]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [x]_\delta$ satisfy that $x_1 \preceq y_1$ and $y_2 \preceq x_2$ with $x_1, x_2 \in [x]_\delta$ and $y_1, y_2 \in [y]_\delta$, then $[x]_\delta = [y]_\delta$.*

Proof. It is straightforwardly deduced from Proposition 22 taking the class $[c]_\delta$ as either the class $[x]_\delta$ or $[y]_\delta$. □

It is easy to see in Figure 5 that $\preceq_\delta$ is not a partial order because of there exists a δ-cycle composed of elements belonging to different equivalence classes, i.e. the δ-cycle $(x_2, x_1, c_2, c_1, y_2, y_1, x_2)$. In order to avoid this problem, every δ-cycle must be contained in one single class, that is, every δ-cycle must be closed in the lattice, as the next result states.

Theorem 24. *Given a lattice $(L, \preceq)$ and a local congruence δ on L, the preorder $\preceq_\delta$ given in Definition 19 is a partial order if and only if every δ-cycle in L is closed.*

Proof. Let us assume that δ is a local congruence on L and that every δ-cycle in L is closed and let us prove that $\preceq_\delta$ is a partial order.

The reflexivity of $\preceq_\delta$ holds in a direct way.

Now, we prove the transitivity. If $[x]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [z]_\delta$ for $[x]_\delta, [y]_\delta, [z]_\delta \in L/\delta$, then there exist two δ-sequences $(x', y_1)_\delta = (x', p_1, \ldots, p_n, y_1)$ and $(y_2, z')_\delta = (y_2, q_1, \ldots, q_m, z')$, with $x' \in [x]_\delta$, $y_1, y_2 \in [y]_\delta$ and $z' \in [z]_\delta$. Hence, there exists a δ-sequence $(x', z')_\delta = (x', p_1, \ldots, p_n, y_1, y_2, q_1, \ldots, q_m, z')$ satisfying the conditions of Definition 19. Thus, $[x]_\delta \preceq_\delta [z]_\delta$ and the relation $\preceq_\delta$ is transitive.

In order to prove that $\preceq_\delta$ is antisymmetric, we assume that $[x]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [x]_\delta$, for some $x, y \in L$. Then there exist $x' \in [x]_\delta$, $y' \in [y]_\delta$, a δ-sequence $(x', y')_\delta = (x', p_1, \ldots, p_n, y')$ and a δ-sequence $(y', x')_\delta = (y', q_1, \ldots, q_m, x')$. Clearly, $(x', p_1, \ldots, y', q_1, \ldots, x')$ is a δ-cycle and since every δ-cycle is closed, we obtain $[x]_\delta = [y]_\delta$. Hence $\preceq_\delta$ is antisymmetric and thus a partial order.

Now, suppose that $\preceq_\delta$ is a partial order, if $(p_0, \ldots, p_n, p_0)$ is a δ-cycle of L (as it is showed in Figure 4) then we have that

$$[p_0]_\delta \circledast_1 [p_1]_\delta \circledast_2 [p_2]_\delta \circledast_3 [p_3]_\delta \circledast_4 \cdots \circledast_{n-2} [p_{n-2}]_\delta \circledast_{n-1} [p_{n-1}]_\delta \circledast_n [p_n]_\delta \circledast_0 [p_0]_\delta$$

where $\circledast_i \in \{=, \preceq_\delta\}$ for all $i \in \{0, \ldots, n\}$. Since the chain begins and ends with the same element, and $\preceq_\delta$ is a partial order, we obtain that the δ-cycle is closed. □

As a consequence, under the assumption of the introduced necessary and sufficient condition, this result allows to order the convex sublattices (classes) obtained after the attribute reduction, which provide a hierarchization among the obtained concepts. The following example shows that this hierarchization does not form a complete lattice.

Example 25. Let $(L, \preceq)$ be the lattice given in the left side of Figure 7 which is isomorphic to a concept lattice obtained from a formal context and δ the local congruence shown in the middle of Figure 7. In this case, the considered local congruence makes that the relation $\preceq_\delta$ be a partial order.

However, the quotient set L/δ ordered with $\preceq_\delta$ does not form a lattice, as it is shown in the right side of Figure 7, because the equivalence classes

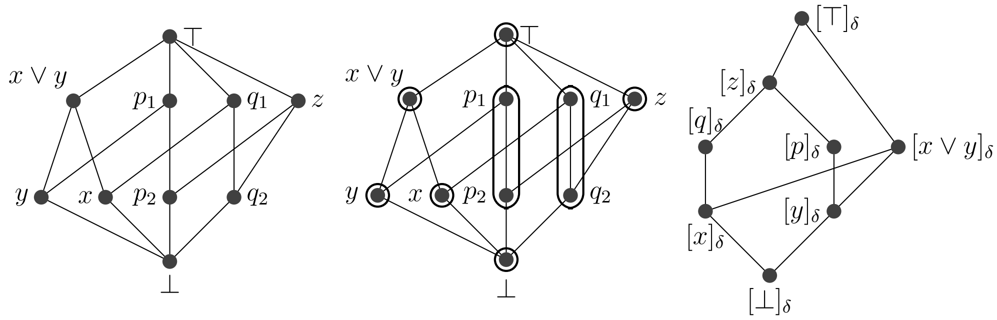

Figure 7: The lattice $(L, \preceq)$ (left), the local congruence δ on L (middle) and its corresponding quotient set $(L/\delta, \preceq_\delta)$ with the ordering relation $\preceq_\delta$ (right).

$[x]_\delta$ and $[y]_\delta$ have not got a supremum, that is, their least upper bound does not exist. □

Therefore, an ordering can be defined on the classes of a local congruence, when every δ-cycle in L is closed, which could not provide a complete lattice, but it is enough to produce a hierarchization among the computed reduced concepts. In order to ensure that a local congruence can always be computed, such as every δ-cycle is closed, more properties of local congruences need to be studied. Specifically, it is important to analyze the relationships among these new congruences.

5. Algebraic structure of local congruences on a lattice

In this section, we study the algebraic structure of the set of all local congruences defined on a lattice. First of all, we show that local congruences can be ordered by using the definition of inclusion of equivalence relations, which is recalled next.

Definition 26. Let ρ_1 and ρ_2 be two equivalence relations on a lattice $(L, \preceq)$. We say that the equivalence relation ρ_1 is included in ρ_2, denoted as $\rho_1 \sqsubseteq \rho_2$, if for every equivalence class $[x]_{\rho_1} \in L/\rho_1$ there exists an equivalence class $[y]_{\rho_2} \in L/\rho_2$ such that $[x]_{\rho_1} \subseteq [y]_{\rho_2}$.

We say that two equivalence relations, ρ_1 and ρ_2, are incomparable if $\rho_1 \not\sqsubseteq \rho_2$ and $\rho_2 \not\sqsubseteq \rho_1$.

From now on, the set of all local congruences on L ordered by the inclusion $\sqsubseteq$ will be denoted as $(\text{LCon } L, \sqsubseteq)$. First of all, we will show that the set (LCon $L, \sqsubseteq$) is a complete lattice, proving that (LCon $L, \sqsubseteq$) is a topped $\sqcap$-structure with a maximum element. In addition, the maximum and the minimun of the complete lattice (LCon $L, \sqsubseteq$) are characterized.

Theorem 27. *Given a lattice $(L, \preceq)$, the set (LCon $L, \sqsubseteq$) is a complete lattice. Moreover, the least and greatest element are given by $\delta_\perp = \{(a, a) \mid a \in L\}$ and $\delta_\top = \{(a, b) \mid a, b \in L\}$, respectively.*

PROOF. Let us assume that $(L, \preceq)$ is a lattice and (LCon $L, \sqsubseteq$) is the set of all local congruences. First of all, we need to prove that (LCon $L, \sqsubseteq$) is a topped $\cap$-structure. Therefore, we consider a non-empty family of local congruence, that is, $\{\delta_i\}_{i\in I} \subseteq$ LCon L where I is a index set.

It is well known that the intersection of equivalence relations is an equivalence relation. Hence, $\bigcap_{i\in I} \delta_i$ is indeed an equivalence relation. Now, we prove that each equivalence class of the intersection is a convex sublattice. Let us consider an equivalence class $Z \in L/(\bigcap_{i\in I} \delta_i)$, hence there exist a family of equivalence classes $\{X_i \in L/\delta_i \mid i \in I\}$ such that $Z = \bigcap_{i\in I} X_i$. If we consider $a, b \in Z$, then we have that $a, b \in X_i$ for all $i \in I$ and, since each X_i is a convex sublattice of L, we have that $a \wedge b, a \vee b \in X_i$ for all $i \in I$. Therefore, $a \wedge b, a \vee b \in \bigcap_{i\in I} X_i = Z$, that is, Z is a sublattice of L. In addition, if $a \preceq b$ and we consider $c \in L$ such that $a \preceq c \preceq b$, then we have that $c \in X_i$ for all $i \in I$ since each X_i is convex. Therefore, $c \in \bigcap_{i\in I} = Z$, that is, Z is also convex. Thus, $\bigcap_{i\in I} \delta_i \in$ LCon L, i.e., LCon L is a $\cap$-structure.

Now, we need to prove that LCon L has a maximum element. It is clear that the equivalence relation on L that relates all elements of L, that is, $\{(a, b) \mid a, b \in L\} = L \times L$, has convex sublattices of L as equivalence classes, hence $\{(a, b) \mid a, b \in L\} = L \times L \in$ LCon L and moreover, we cannot find another local congruence that contains it. Therefore, $\{(a, b) \mid a, b \in L\} = L \times L$ is the greatest local congruence and we denote it as $\delta_\top$. Thus, the set (LCon $L, \sqsubseteq$) is a complete lattice.

In addition, it is clear that the least local congruence is the equivalence relation on L that only relates each elements of L to itself, that is, $\delta_\perp = \{(a, a) \mid a \in L\}$. □

Next definition shows the notion of principal local congruence, which is the least local congruence that can be defined from two given elements of a lattice.

Definition 28. Given a pair of elements $(a,b) \in L \times L$, the *principal local congruence generated by* (a,b), denoted as $\delta_{(a,b)}$, is the least local congruence that contains the elements a and b in the same equivalence class, that is

$$\delta_{(a,b)} = \bigwedge \{\delta \in \text{LCon } L \mid (a,b) \in \delta\}$$

Note that, for every pair of elements $(a,b) \in L \times L$, the principal local congruence $\delta_{(a,b)}$ always exists since the set $(\text{LCon } L, \sqsubseteq)$ is a complete lattice.

Finally, the last theorem generalizes the characterization of congruences in terms of principal congruences (recalled in Lemma 6) for local congruences, considering an arbitrary equivalence relation.

Theorem 29. *Given a lattice* $(L, \preceq)$ *and an equivalence relation* ρ*, the least local congruence containing* ρ *is*

$$\delta_\rho = \bigvee \{\delta_{(a,b)} \mid (a,b) \in \rho\}$$

PROOF. Let us assume that δ_ρ is the least local congruence containing an equivalence relation ρ and let us prove that δ_ρ is the least upper bound of the set $S = \{\delta_{(a,b)} \mid (a,b) \in \rho\}$. Due to $\rho \sqsubseteq \delta_\rho$, it is clear that $S \subseteq \{\delta_{(c,d)} \mid (c,d) \in \delta_\rho\}$ and, by Proposition 30, we have that $\delta_\rho = \bigvee\{\delta_{(c,d)} \mid (c,d) \in \delta_\rho\}$. Hence δ_ρ is an upper bound for S. Now, let us assume that δ'_ρ is an upper bound for S, which means that for all $(a,b) \in \rho$ then $\delta_{(a,b)} \sqsubseteq \delta'_\rho$. Therefore, by the supremum property we have that

$$\delta_\rho = \bigvee \{\delta_{(a,b)} \mid (a,b) \in \rho\} \sqsubseteq \delta'_\rho$$

which finishes the proof. □

In particular, the previous result is also satisfied when we consider a local congruence instead of an arbitrary equivalence relation.

Corollary 30. *Let* $(L, \preceq)$ *be a lattice and let* δ *a local congruence of* $(LCon\ L, \sqsubseteq)$. *Then*

$$\delta = \bigvee \{\delta_{(a,b)} \mid (a,b) \in \delta\}.$$

PROOF. Straightforwardly from Theorem 29, considering a local congruence δ as the equivalence relation ρ. □

Note that this result will be very important in the reduction procedure in order to obtain a local congruence δ satisfying that $(L/\delta, \preceq_\delta)$ is a partial ordered set (poset), as we will show in the next section.

6. Reduction mechanism of concept lattices

This section will introduce an attribute reduction mechanism focused on grouping concepts in convex sublattices, having a hierarchy in form of a poset, which is equivalent by Theorem 24 to computing a local congruence with all δ-cycle in L being closed. In order to fulfill this last requirement we will use the following procedure from an arbitrary local congruence. Given a lattice $(L, \preceq)$ and a local congruence δ on L, if every δ-cycle in L is closed, then we already have that $(L/\delta, \preceq_\delta)$ is a poset. Otherwise, we can define an equivalence relation ρ on L/δ as

$$\rho_\delta = \{([x]_\delta, [y]_\delta) \in L/\delta \times L/\delta \mid [x]_\delta \preceq_\delta [y]_\delta \text{ and } [y]_\delta \preceq_\delta [x]_\delta\} \tag{4}$$

If there are two different equivalence classes $[x]_\delta$, $[y]_\delta$ such that $[x]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [x]_\delta$, this means that there is a δ-cycle, $(x', x')_\delta$ or $(y', y')_\delta$ for some $x' \in [x]_\delta$, $y' \in [y]_\delta$. Therefore, the equivalence relation ρ_δ groups all the equivalence classes that contain elements in the δ-cycle in a unique equivalence class providing a new partition of L.

However, the equivalence relation ρ_δ may not be a local congruence. Since clearly $\delta \sqsubseteq \rho_\delta$, by Theorem 29, we can find the least local congruence $\bar{\delta}$ that contains the equivalence relation ρ_δ, that is, $\delta \sqsubseteq \rho_\delta \sqsubseteq \bar{\delta}$. Hence, every $\bar{\delta}$-cycle in L is closed and, by Theorem 24, $\preceq_{\bar{\delta}}$ is a partial order on the quotient set $L/\bar{\delta}$.

This procedure to ensure the ordering between the classes will be incorporate in the procedure to reduce concept lattices by local congruences, which is summarized in the following steps:

Notice that, the set D in Algorithm 1 can be given from any reduction mechanism. For example, it can be a rough set reduct [8, 7]. Moreover, observe that the relation ρ_D was defined in the classical case in Proposition 9 and in the fuzzy case in Proposition 11.

This previous mechanism provides the desired reduction, as the following result shows.

Proposition 31. *Given a concept lattice $\mathcal{C}(A, B, R)$ and a subset of attributes $D \subseteq A$, then Algorithm 1 provides the least local congruence δ containing the induced relation ρ_D and $(\mathcal{C}(A, B, R)/\delta, \preceq_\delta)$ is a poset.*

Proof. Let us assume that we have a concept lattice $\mathcal{C}(A, B, R)$ and a partition of $\mathcal{C}(A, B, R)$ induced by an attribute reduction provided by $D \subseteq A$.

Algorithm 1: Reducing concept lattices by local congruences

input : $\mathcal{C}(A, B, R)$, $D \subseteq A$
output: δ

Obtain the relation ρ_D associated with the attribute reduction given by D;
Compute the least local congruence δ_D containing ρ_D;
if *every* δ_D*-cycle is closed,* **then**
    $\delta = \delta_D$
else
    Compute ρ_{δ_D} by Equation (4);
    if ρ *is a local congruence* **then**
        $\delta = \rho_{\delta_D}$
    else
        Obtain the least local congruence δ_ρ such that $\delta_D \sqsubseteq \rho_{\delta_D} \sqsubseteq \delta_\rho$;
        $\delta = \delta_\rho$
return δ

The starting point of the procedure (Line 1) is the computation of the equivalence relation associated with the attribute reduction given by the subset D, which is denoted as ρ_D.

In Line 2, by Theorem 29, we obtain the least local congruence containing ρ_D, which is denoted as δ_D. Hence, in particular, $\rho_D \sqsubseteq \delta_D$. From this local congruence, the relation $\preceq_{\delta_D}$ defined as in Definition 19 is a preorder. By Theorem 24, if every δ_D-cycle is closed (checked in Line 3), then $(\mathcal{C}(A, B, R)/\delta, \preceq_\delta)$ is a poset, that is, the required relation δ is δ_D (Line 4).

Otherwise, $\preceq_{\delta_D}$ is only a preorder and we consider the new equivalence relation ρ_{δ_D} defined in Equation 4. As a consequence, we have that $\delta_D \sqsubseteq \rho_{\delta_D}$. If ρ_{δ_D} is a local congruence, by the definition of ρ_{δ_D}, we have that every ρ_{δ_D}-cycle is closed and, according to Theorem 24, $\preceq_{\rho_{\delta_D}}$ is a partial order on $\mathcal{C}(A, B, R)/\rho_{\delta_D}$. In this case, $\delta = \rho_{\delta_D}$ is the least local congruence we are interested in (Lines 6-8). Otherwise, from Theorem 29, in Line 10 we obtain the least local congruence δ_ρ containing to ρ_{δ_D}, such that $\rho_{\delta_D} \sqsubseteq \delta_\rho$. Therefore, we have that every δ_ρ-cycle is closed by the definition of ρ_{δ_D}, and by Theorem 24 we obtain that $\preceq_{\delta_\rho}$ is a partial order on $\mathcal{C}(A, B, R)/\delta_\rho$. Thus, $\delta = \delta_\rho$ is the least local congruence we are looking for.

Consequently, from the procedure we obtain that $(\mathcal{C}(A, B, R)/\delta, \preceq_\delta)$ is a poset where δ is the least local congruence containing ρ_D. □

In the next example we will show the procedure described above.

Example 32. Let us consider a context (A, B, R) and a subset of attributes $D \subseteq A$ such that after the reduction process we obtain the induced partition of the concept lattice displayed in the left side of Figure 8. Thus, we consider the local congruence δ_D displayed in the right side of Figure 8, it is easy to check that δ_D is indeed a local congruence and the least one containing the induced partition.

Considering the relation $\preceq_{\delta_D}$ given as in Definition 19, we can note that the δ_D-sequence, $(p_0, p_0)_{\delta_D} = (p_0, p_5, p_1, p_7, p_2, p_3, p_0)$, is in fact a δ_D-cycle in the lattice and it is not closed. Therefore, we define the equivalence relation $\rho = \{([x]_{\delta_D}, [y]_{\delta_D}) \in L/\delta_D \times L/\delta_D \mid [x]_{\delta_D} \preceq_{\delta_D} [y]_{\delta_D} \text{ and } [y]_{\delta_D} \preceq_{\delta_D} [x]_{\delta_D}\}$. The new partition of L provided by the equivalence relation ρ is shown in the left side of Figure 9. We can observe that the equivalence relation ρ groups the classes of L/δ_D that contain elements in the δ_D-cycle into a single equivalence class. Moreover, it is also easy to observe that $\delta_D \sqsubseteq \rho$.

Now, we have to verify if the equivalence relation ρ is a local congruence, but we can observe that it is not, since the equivalence class that contains

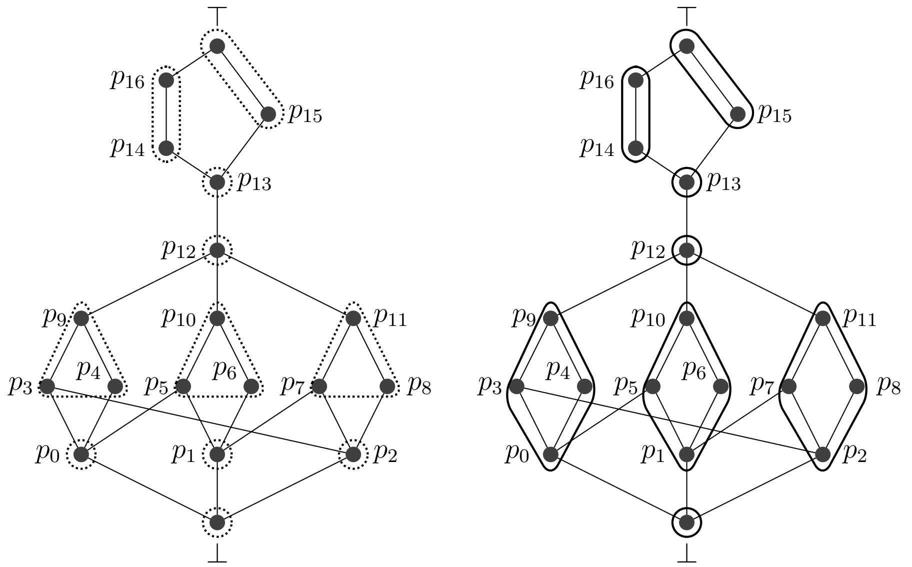

Figure 8: Partition induced ρ_D (left) and local congruence δ_D (right) of Example 32.

the δ_D-cycle is not a (convex) sublattice of L. Thus, we must find the least local congruence δ_ρ that contains the equivalence relation ρ.

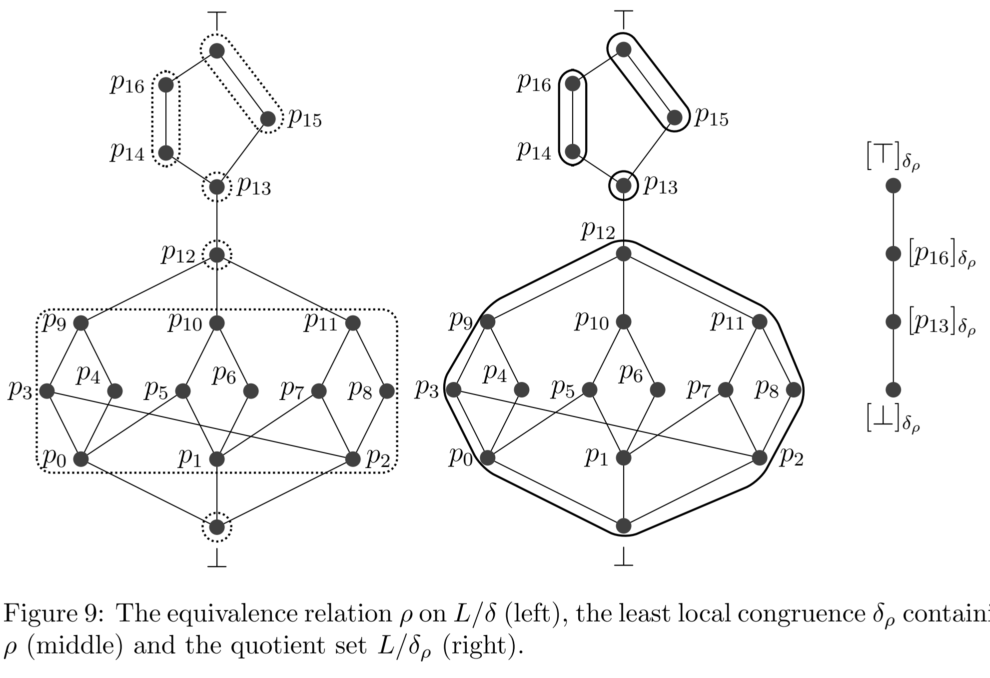

Figure 9: The equivalence relation ρ on L/δ (left), the least local congruence δ_ρ containing ρ (middle) and the quotient set L/δ_ρ (right).

In this case, the least local congruence that satisfies $\delta_D \sqsubseteq \rho \sqsubseteq \delta_\rho$ is the local congruence shown in the middle of Figure 9. Hence, by Theorem 24, we have that $\preceq_{\delta_\rho}$ is a partial order on L/δ_ρ and the elements of the corresponding quotient set can be ranked. The ordered set $(L/\delta_\rho, \preceq_{\delta_\rho})$, is displayed in the right side of Figure 9. It is important to note that the local congruence that we have finally obtained is not a congruence because the least congruence, containing the equivalence relation ρ, should include $p_{13}, p_{14}, p_{15}, p_{16}$ and $\top$ in the same class, in order to satisfy the quadrilateral-closed property. □

Now, we apply the proposed mechanism to reduce a concept lattice in a fuzzy formal concept framework. Specifically, the following example considers a fuzzy formal context studied in [7].

Example 33. The considered framework is $(L, L, L, \&_G^*)$, where the lattice $L = \{0, 0.5, 1\}$ and $\&_G^*$ is the discretization of the Gödel conjunctor defined on L. It is also considered a fuzzy context (A, B, R, σ), composed of three objects, $B = \{b_1, b_2, b_3\}$, four attributes $A = \{a_1, a_2, a_3, a_4\}$, the relation R shown in Table 2, and the mapping σ constantly $\&_G^*$. All concepts of this fuzzy context are listed in Figure 10, where the corresponding concept lattice is illustrated as well.

R	b_1	b_2	b_3
a_1	1	0	0
a_2	0	0.5	0
a_3	0	0	1
a_4	0	0.5	1

Table 2: Fuzzy relation R of Example 33.

In [7], authors obtained four different reducts to reduce the concept lattice. In this example, we will consider one of these reducts, specifically $D_1 = \{a_1, a_2\}$, to compute a local congruence of the reduced concept lattice obtained from this reduct.

The partition induced by D_1 is given in the left side of Figure 11, and the corresponding reduced concept lattice is depicted in its right side. In this case, the least local congruence containing this partition is the partition itself, since each equivalence class is a convex sublattice of the original concept lattice. Moreover, this local congruence is not a congruence because

C_i	Extent			Intent			
	b_1	b_2	b_3	a_1	a_2	a_3	a_4
0	0	0	0	1	1	1	1
1	1	0	0	1	0	0	0
2	0	0.5	0	0	1	0	1
3	0	0	1	0	0	1	1
4	1	1	1	0	0	0	0
5	0	1	0	0	0.5	0	0.5
6	0	0.5	1	0	0	0	1
7	0	1	1	0	0	0	0.5

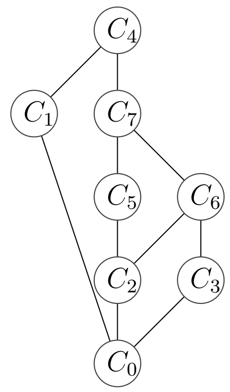

Figure 10: Fuzzy concepts (left) and concept lattice (right) of the context associated with Table 2.

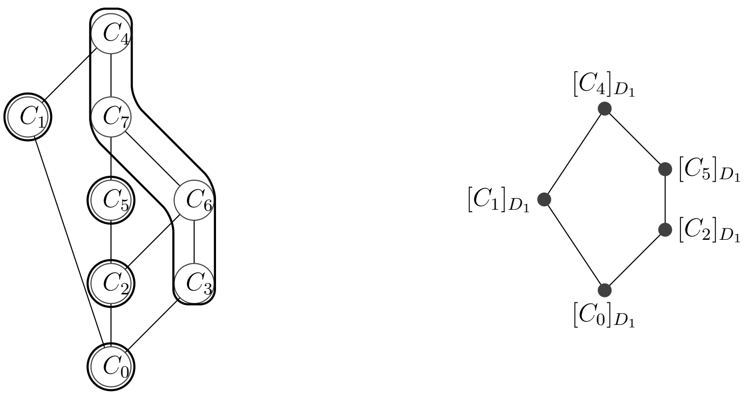

Figure 11: Partition induced by the reduction (left) and concept lattice of the reduced context (right) considering the reduct D_1.

it is not quadrilateral-closed, for example, C_0, C_2 and C_3, C_6 are opposite sides of the quadrilateral $\langle C_0, C_2; C_3, C_6\rangle$, the concepts C_3 and C_6 belong to one equivalence class, but C_0 and C_2 belong to different equivalence classes. Indeed, the least congruence containing the partition induced by the reduction of D_1 is the congruence with only one class containing all concepts. Thus, also in the fuzzy framework, local congruences offer more suitable reductions than the ones given by congruences. □

Therefore, the proposed reduction mechanism based on local congruences minimizes the amount of lost information with respect to the use of congruences, clustering the concepts in convex sublattices and forming a hierarchy among them.

7. Conclusions and future work

In this work, we have introduced a weaker notion of congruence, which has been called local congruence. We have analyzed how the elements of the quotient set generated by a local congruence can be ordered. Furthermore, we have proven that the algebraic structure of the set of local congruences is a complete lattice. We have also shown a characterization of local congruences in terms of its principal local congruences, as well as an extension of this characterization by considering any arbitrary equivalence relation. As a consequence, a procedure for computing the least local congruence containing a given equivalence relation has been presented. From this study, we have presented a new mechanism to reduce (fuzzy) concept lattices based on the notion of local congruence. Considering this reduction mechanism, we obtain a partition of the concepts of the original concept lattice satisfying that each equivalence class has the structure of a convex sublattice of the original concept lattice. In addition, we have shown that the consideration of local congruences to reduce concept lattices is more suitable than the consideration of congruences since a smaller amount of information is lost during the reduction process.

In the near future, more properties of the introduced procedure will be studied. For example, due to this reduction modifies the original partition given by the attribute reduction, it is important to analyze how it alters the formal context. In addition, we are interested in studying how an optimal reduct can be selected and the influence that this selection has on the complementary local congruence. Another important goal will be to apply this reduction procedure in real databases. Specifically, we would like to analyze the potential of the presented reduction mechanism in databases related to digital forensic analysis, in which we are leading the COST Action: DIGital FORensics: evidence Analysis via intelligent Systems and Practices (DigForASP).

References

[1] C. Alcalde and A. Burusco. Study of the relevance of objects and attributes of l-fuzzy contexts using overlap indexes. In J. Medina, M. Ojeda-Aciego, J. L. Verdegay, D. A. Pelta, I. P. Cabrera, B. Bouchon-Meunier, and R. R. Yager, editors, *Information Processing and Management of Uncertainty in Knowledge-Based Systems. Theory and Foundations*, pages 537–548, Cham, 2018. Springer International Publishing.

[2] L. Antoni, M. E. Cornejo, J. Medina, and E. Ramirez. Attribute classification and reduct computation in multi-adjoint concept lattices. *IEEE Transactions on Fuzzy Systems*, pages 1–1, 2020.

[3] L. Antoni, S. Krajči, and O. Krídlo. Constraint heterogeneous concept lattices and concept lattices with heterogeneous hedges. *Fuzzy Sets and Systems*, 303:21 – 37, 2016.

[4] L. Antoni, S. Krajci, O. Kridlo, B. Macek, and L. Pisková. On heterogeneous formal contexts. *Fuzzy Sets and Systems*, 234:22–33, 2014.

[5] R. G. Aragón, J. Medina, and E. Ramírez-Poussa. Weaken the congruence notion to reduce concept lattices. In *European Symposium on Computational Intelligence and Mathematics (ESCIM2019)*, pages 45–46, 2019.

[6] R. Bělohlávek, T. Funiokova, and V. Vychodil. Galois connections with hedges. In *Proc. IFSA 2005 World Congress*, volume II, pages 1250–1255. Springer, 2005.

[7] M. J. Benítez-Caballero, J. Medina, E. Ramírez-Poussa, and D. Ślęzak. Rough-set-driven approach for attribute reduction in fuzzy formal concept analysis. *Fuzzy Sets and Systems*, 391:117 – 138, 2020. Computer Science.

[8] M. J. Benítez-Caballero, J. Medina, E. Ramírez-Poussa, and D. Ślęzak. A computational procedure for variable selection preserving different initial conditions. *International Journal of Computer Mathematics*, 97(1-2):387–404, 2020.

[9] T. S. Blyth. *Lattices and Ordered Algebraic Structures*. Springer, 2005.

[10] A. Burusco and R. Fuentes-González. Construction of the L-fuzzy concept lattice. *Fuzzy Sets and Systems*, 97(1):109–114, 1998.

[11] J. Chen, J. Mi, B. Xie, and Y. Lin. A fast attribute reduction method for large formal decision contexts. *International Journal of Approximate Reasoning*, 106:1 – 17, 2019.

[12] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Attribute reduction in multi-adjoint concept lattices. *Information Sciences*, 294:41 – 56, 2015.

[13] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Attribute and size reduction mechanisms in multi-adjoint concept lattices. *Journal of Computational and Applied Mathematics*, 318:388 – 402, 2017.

[14] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Characterizing reducts in multi-adjoint concept lattices. *Information Sciences*, 422:364 – 376, 2018.

[15] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002.

[16] B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundation*. Springer Verlag, 1999.

[17] G. Grätzer. *General Lattice Theory*. Birkhäuser Basel, 2nd edition, 2007.

[18] G. Grätzer. *Universal Algebra*. Springer-Verlag New York, 2nd edition, 2008.

[19] J. Konecny and P. Krajča. On attribute reduction in concept lattices: The polynomial time discernibility matrix-based method becomes the cr-method. *Information Sciences*, 491:48 – 62, 2019.

[20] J.-Y. Li, X. Wang, W.-Z. Wu, and Y.-H. Xu. Attribute reduction in inconsistent formal decision contexts based on congruence relations. *International Journal of Machine Learning and Cybernetics*, 8(1):81–94, 2017.

[21] J. Medina. Multi-adjoint property-oriented and object-oriented concept lattices. *Information Sciences*, 190:95–106, 2012.

[22] J. Medina. Relating attribute reduction in formal, object-oriented and property-oriented concept lattices. *Computers & Mathematics with Applications*, 64(6):1992–2002, 2012.

[23] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.

[24] R. Ren and L. Wei. The attribute reductions of three-way concept lattices. *Know.-Based Syst.*, 99(C):92–102, May 2016.

[25] M.-W. Shao and K.-W. Li. Attribute reduction in generalized one-sided formal contexts. *Information Sciences*, 378:317 – 327, 2017.

[26] J. Viaud, K. Bertet, C. Demko, and R. Missaoui. The reverse doubling construction. In *2015 7th International Joint Conference on Knowledge Discovery, Knowledge Engineering and Knowledge Management (IC3K)*, volume 01, pages 350–357, Nov 2015.

[27] J.-F. Viaud, K. Bertet, C. Demko, and R. Missaoui. Subdirect Decomposition of Contexts into Subdirectly Irreducible Factors. In *International Conference on Formal Concept Analysis ICFCA2015*, Nerja, Spain, June 2015.

[28] J.-F. Viaud, K. Bertet, R. Missaoui, and C. Demko. Using congruence relations to extract knowledge from concept lattices. *Discrete Applied Mathematics*, 249:135 – 150, 2018.

[29] G. Vojvodić and B. Šešelja. On the lattice of weak congruence relations. *algebra universalis*, 25(1):121–130, Dec 1988.

[30] B. Šešelja, V. Stepanović, and A. Tepavčević. Representation of lattices by fuzzy weak congruence relations. *Fuzzy Sets and Systems*, 260:97 – 109, 2015.

[31] B. Šešelja and A. Tepavčević. A note on atomistic weak congruence lattices. *Discrete Mathematics*, 308(10):2054 – 2057, 2008.

[32] A. Walendziak. Weak congruences of an algebra with the cep and the wcip. *Czechoslovak Mathematical Journal*, 52(1):117–127, Mar 2002.