On the solvability of bipolar max-product fuzzy relation equations with the product negation

M. Eugenia Cornejo, David Lobo, Jesús Medina

Department of Mathematics, University of Cádiz. Spain
`{mariaeugenia.cornejo,david.lobo,jesus.medina}@uca.es`

Abstract

This paper studies the solvability of the max-product fuzzy relation equations in which a negation operator is considered. Specifically, the residuated negation of the product t-norm has been introduced in these equations in order to increase the flexibility of the standard fuzzy relation equations introduced by Sanchez in 1976. The solvability and the set of solutions of these bipolar equations have been studied in different scenarios, depending on the considered number of variables and equations.

Keywords: Bipolar fuzzy relation equations, max-product composition, negation operator.

1. Introduction

Fuzzy relation equations (FREs) based on different max-t-norm compositions have become an extremely important research topic both from the theoretical and applicative perspective. First of all, Sanchez introduced these equations by using the max-min composition for simulating the relationship between cause and effect in medical diagnosis problems [50, 51]. Later, Pedrycz studied max-product fuzzy relation equations showing the applicative aspects of such equations in systems analysis, decision making and arithmetic of fuzzy numbers [45]. Di Nola et al. provided a detailed study on max-continuous t-norm FREs and their applications to knowledge engineering in [13]. Thenceforward, many researchers have continued investigating different procedures to solve FREs defined with either the max-min composition [6, 7, 24, 59], the max-product composition [5, 34, 36, 47], the max-Archimedean t-norm composition [31, 55, 56] or other different compositions [2, 3, 12, 14, 15, 16, 23, 37, 38, 46, 48].

As well as developing technical results on the resolution of FREs, the versatility of the applications of such equations in fuzzy sets theory has been increased. For instance, FREs play an important role in the treatment of images [20, 21, 35, 42, 43, 44]. Specifically, max-Lukasiewicz FREs are considered for images and videos compression and decompression in [35]. Fuzzy relation equations are also helpful to model the fuzzy inference systems considered in fuzzy control [8] and to represent restrictions in optimization problems [28, 29, 39, 58]. In particular, max-product fuzzy relation equations are used for optimizing wireless communication management models in [58]. FREs have also been related to other mathematical frameworks, such as fuzzy formal concept analysis [11, 14, 15].

Many of the applications previously mentioned require that the variables involved in fuzzy relation equations show a bipolar character. That is, these applications need equations containing unknown variables together with their logical negations simultaneously. For example, let us consider the equation $0.7 * x = b$, where $*$ is the product t-norm, x indicates the level of infection one patient has in the throat and b the level of headache. This relation between infection and headache is modeled by the coefficient 0.7. If the doctor knows that the patient has an initial infection, with truth degree 0.5, then we have that the patient has headache with truth degree: 0.35. It is clear that this equation is a simple equation from the whole set of equations for simulating the considered knowledge system. Suppose that the patient has not infection and has headache, with truth value 0.8. In this case, the equation $0.7 * x = b$ does not work. However, if we consider the product negation operator defined as $n(x) = 1$, if $x = 0$, and $n(x) = 0$, otherwise, the previous equation can be conveniently reformulated obtaining the following one:

$$(0.7 * x) \vee (0.8 * n(x)) = b$$

which is solvable and captures the cases when the patient has the disease but does not have the symptoms. This new type of fuzzy relation equation, in which a negation operator is considered, is called bipolar fuzzy relation equation. Notice that the use of the product negation increases the flexibility of the usual (unipolar) fuzzy relation equations and, at the same time, the complexity in their resolution, which becomes even more difficult because the product negation is not involutive.

Bipolar fuzzy relation equations have been already employed usefully in optimization problems [18, 22, 27, 33, 60]. To the best of our knowledge, the specific literature on the resolution of bipolar fuzzy relation equations is

really limited [30, 32]. Moreover, these papers are focused on the resolution of bipolar fuzzy relation equations based on max-min composition with the standard negation. Taking into account these considerations, we can ensure that the study on the solvability of bipolar FREs is an incipient research topic which has not been enough investigated so far.

This paper aims to provide under what conditions the solvability of bipolar max-product fuzzy relation equations with the product negation is guaranteed, continuing the initial study about this topic presented by the authors in [9]. First of all, we will include the preliminary notions related to the calculus operators which will be used in bipolar max-product fuzzy relation equations. In the following, we will introduce the notion of bipolar max-product FRE with the product negation and a unique unknown variable. Simple scenarios will be analyzed and a characterization on the solvability of such equations will be given. Providing sufficient and necessary conditions to ensure when bipolar max-product FREs, with the product negation and different unknown variables, are solvable will be our next goal. Moreover, we will include different properties associated with the existence of the greatest/least solution or a finite number of maximal/minimal solutions for these last equations. We will also deal with the resolution of bipolar max-product FRE systems containing the product negation. It is also convenient to mention that the results obtained in this paper differ markedly from those presented in [30, 32], since the behaviour of the minimum t-norm and the product t-norm is clearly different. Some conclusions and prospects for future work are included at the end of the paper.

2. Preliminary notions

This section introduces basic notions and examples associated with the operators that we will use to make the computations in this paper.

Triangular norms are interesting operators which play an important role in different fields of mathematics such as probabilistic metric spaces [53, 54], decision making [17, 19], statistics [40], fuzzy sets theory and its applications [1, 57]. A detailed survey on the basic analytical and algebraic properties of triangular norms can be found in [26]. Below, we will include the formal definition of a triangular norm.

Definition 1. A binary operation $T \colon [0, 1] \times [0, 1] \to [0, 1]$ is a *triangular norm* (*t-norm*) if the following properties are satisfied, for all $x, y, z \in [0, 1]$:

(1) $T(x, y) = T(y, x)$ (commutativity);

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(2) If $x \le y$ then $T(x, z) \le T(y, z)$ (monotonicity);

(3) $T(x, 1) = x$ (neutral element);

(4) $T(x, T(y, z)) = T(T(x, y), z)$ (asociativity).

T-norms are operators frequently used in applicative examples [25, 41, 52] together with their residuated implications.

Definition 2. Given a t-norm $T \colon [0, 1] \times [0, 1] \to [0, 1]$, if there exists a binary operation $\leftarrow_T \colon [0, 1] \times [0, 1] \to [0, 1]$ order-preserving on the first argument and order-reversing on the second argument verifying the equivalence:

$$T(x, y) \le z \text{ if and only if } x \le z \leftarrow_T y \tag{1}$$

then we say that $\leftarrow_T$ is a *residuated implication of* T. The pair $(T, \leftarrow_T)$ is called *residuated pair* and Equivalence (1) is called adjoint property.

The residuated pairs of the most commonly used t-norms in the literature will be recalled in the following example.

Example 3. Example of residuated pairs are the Gödel, product and Łukasiewicz t-norms together with their corresponding residuated implications, which are defined on $[0, 1]$, as you can see below:

$$T_{\mathrm{P}}(x, y) = x \cdot y \qquad\qquad z \leftarrow_{\mathrm{P}} x = \min\{1, z/x\}$$

$$T_{\mathrm{G}}(x, y) = \min\{x, y\} \qquad\qquad z \leftarrow_{\mathrm{G}} x = \begin{cases} 1 & \text{if } x \le z \\ z & \text{otherwise} \end{cases}$$

$$T_{\mathrm{L}}(x, y) = \max\{0, x + y - 1\} \qquad z \leftarrow_{\mathrm{L}} x = \min\{1, 1 - x + z\}$$

$\square$

The residuated negation will be another fuzzy operator which will play a key role in this paper. These residuated negation operators are defined from the implication of a residuated pair. More information about this kind of negations can be found in [10].

Definition 4. Given an adjoint pair $(T, \leftarrow_T)$ defined on $[0, 1]$, the mapping $n_T \colon [0, 1] \to [0, 1]$ defined as $n_T(x) = 0 \leftarrow_T x$, for all $x \in [0, 1]$, is called *residuated negation*. We say that n_T is a *strong (or involutive) residuated negation* if the equality $x = n_T(n_T(x))$ holds for all $x \in [0, 1]$.

In the following example, we will introduce the residuated negations defined from the Gödel, product and Łukasiewicz implications.

Example 5. Taking into account the definition of the operators introduced in Example 3, it is easy to see that the residuated negation associated with the Łukasiewicz implication is

$$n_{\mathrm{L}}(x) \;\; = \;\; 1 - x$$

for all $x \in [0, 1]$, which is involutive and commonly known in the literature as the standard negation. On the other hand, the Gödel implication and the product implication coincide for all $x \in [0, 1]$:

$$n_{\mathrm{P}}(x) \;\; = \;\; n_{\mathrm{G}}(x) \;\; = \;\; \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

which are not involutive. This negation operator will be called *product negation.* $\qquad\square$

In order to make the paper self-contained, we present the Intermediate Value Theorem [4] (english translation in [49]), which will be used in different results later on.

Theorem 6 (Intermediate Value Theorem). *Let $f \colon [a, b] \to \mathbb{R}$ be a continuous function and $c \in \mathbb{R}$ be any number between $f(a)$ and $f(b)$ inclusive. Then, there exists $x \in [a, b]$ such that $f(x) = c$.*

From now on, we will focus on solving bipolar fuzzy relation equations based on the max-product t-norm composition. Our study will analyze the resolution of these equations when the considered negation operator is the non-involutive product negation.

3. Bipolar max-product FREs with the product negation

This section introduces an interesting study on how to solve bipolar max-product fuzzy relation equations with the product negation. The results will

be progressively presented according to their complexity, that is: (1) we will firstly analyze when a bipolar max-product FRE with the product negation and a unique unknown variable has solution; (2) hereunder, we will show under what conditions bipolar max-product FREs with different unknown variables are solvable and have a greatest (least, respectively) solution or a finite number of maximal (minimal, respectively) solutions; (3) we will finish this section with a characterization on the solvability of bipolar max-product FRE systems.

3.1. Solving bipolar max-product FREs with one unknown variable

The most simple scenario which can be considered in this paper is the study of a bipolar max-product FRE containing a unique unknown variable. In order to provide a characterization on the solvability of these equations, we need to include the following formal definition.

Definition 7. Let $a^+, a^-, b \in [0, 1]$, x be an unknown variable in $[0, 1]$, $*$ be the product t-norm, $\vee$ be the maximum operator and n_P be the product negation. Equation (2) is called *bipolar max-product fuzzy relation equation with the product negation.*

$$(a^+ * x) \vee (a^- * n_P(x)) = b \qquad (2)$$

Before presenting under what conditions Equation (2) is solvable, we will analyze its solvability when either a^+, a^- or b is equal to 0. The following result characterizes the solvability of Equation (2) in the previous three different cases.

Theorem 8. Let $a^+, a^-, b \in [0, 1]$ and x be an unknown variable belonging to $[0, 1]$. Then, the following statements hold:

(1) If $a^+ = 0$, then Equation (2) is solvable if and only if $b = 0$ or $a^- = b$.

(2) If $a^- = 0$, then Equation (2) is solvable if and only if $b \leq a^+$.

(3) If $b = 0$, then Equation (2) is solvable if and only if $a^+ = 0$ or $a^- = 0$.

PROOF. *In order to demonstrate Statement (1), suppose that $a^+ = 0$. In that case, we obtain that Equation (2) becomes into $a^- * n_P(x) = b$. Taking into account the definition of the product negation, we can obtain that $b = 0$ if and only if any $x \in (0, 1]$ is a solution of Equation (2). In addition,*

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the equality $a^- = b$ holds if and only if $x = 0$ is a solution Equation (2). Therefore, Statement (1) is verified.

Now, assume that $a^- = 0$. As a consequence, Equation (2) is given by $a^+ * x = b$. Due to the product t-norm is a continuous order-preserving mapping on $[0, 1]$ and the equalities $a^+ * 0 = 0$ and $a^+ * 1 = a^+$ hold, we can assert by the Bolzano's theorem that there exists $x \in [0, 1]$ such that $a^+ * x = b$ if and only if $b \leq a^+$. Hence, we obtain that Statement (2) is satisfied.

Finally, if $b = 0$, then Equation (2) becomes into $(a^+ * x) \vee (a^- * n_P(x)) = 0$, which is solvable if and only if $a^+ * x = 0$ and $a^- * n_P(x) = 0$. According to the definition of the product negation, we can deduce that Equation (2) is solvable if and only if $a^+ = 0$ or $a^- = 0$. As a result, we conclude that Statement (3) holds. □

After studying the most trivial cases which can be given in a bipolar max-product FRE, we will continue our research assuming that each known variable appearing in bipolar max-product FREs is different from zero.

A characterization on the solvability of bipolar max-product FREs with the product negation and a unique unknown variable is provided by the next theorem.

Theorem 9. *Let $a^+, a^-, b \in (0, 1]$ and x be an unknown variable belonging to $[0, 1]$. The bipolar max-product FRE given by Equation (2) is solvable if and only if $a^- = b$ or $b \leq a^+$. In this case, at most two solutions exists: $x = 0$ or/and b/a^+, which are related to the conditions $a^- = b$ and $b \leq a^+$, respectively.*

PROOF. First of all, we will prove that if $a^- = b$ or $a^+ \geq b$ then Equation (2) has solution. On the one hand, if $a^- = b$ then $x = 0$ is clearly a solution of Equation (2). On the other hand, we suppose that $a^+ \geq b$. Now, we consider the mapping $f: [0, 1] \to [0, 1]$ defined as $f(x) = a^+ * x$, which is continuous. Since $b \in (0, 1]$ is a number between $f(0) = a^+ * 0 = 0$ and $f(1) = a^+ * 1 = a^+$, applying the Intermediate Value Theorem (Theorem 6), there exists $y \in [0, 1]$ such that $f(y) = a^+ * y = b$. In addition, we can ensure that the previous value $y \in [0, 1]$ is unique since f is a strictly order-preserving mapping. Specifically, by the definition of the product implication, $y = b \leftarrow_P a^+$. In the following, we will demonstrate that

$y = b \leftarrow_P a^+$ is a solution of Equation (2):

$$
\begin{aligned}
(a^+ * y) \vee (a^- * n_P(y)) &= (a^+ * (b \leftarrow_P a^+)) \vee (a^- * n_P(b \leftarrow_P a^+)) \\
&= (a^+ * b/a^+) \vee (a^- * n_P(b/a^+)) \\
&= b \vee (a^- * 0) = b \vee 0 = b
\end{aligned}
$$

being $b \neq 0$ by hypothesis. Consequently, we obtain that Equation (2) is solvable.

In order to prove the counterpart, we will suppose that $a^- \neq b$ and $a^+ < b$, and we will prove that Equation (2) is not solvable. Clearly, for each $x \in [0,1]$, we have that the inequality $a^+ * x < b$ holds. Therefore, Equation (2) is solvable if and only if there exists $x \in [0,1]$ such that $a^- * n_P(x) = b$. However, this last condition is not verified, since n_P can only take either the value 0 or the value 1 and, by hypothesis, $a^- \neq b$ and $b \neq 0$. Hence, we can conclude that Equation (2) is not solvable.

Finally, when Equation (2) is solvable, it has at most two solutions which are given by $x = 0$ and $x = b \leftarrow_P a^+$. It is easy to see that $x = b \leftarrow_P a^+ = b/a^+$ is the greatest solution when both $x = 0$ and $x = b \leftarrow_P a^+$ solve Equation (2). □

An illustrative example will be shown in order to clarify the previous theorem.

Example 10. A simple bipolar max-product FRE is provided by Equation (3) and we will apply Theorem 9 to check whether such equation is solvable or not.

$$
(0.5 * x) \vee (0.2 * n_P(x)) = 0.3 \tag{3}
$$

Observe that, $0.2 \neq 0.3$ and $0.5 \geq 0.3$. Therefore, we can ensure that the hypothesis required in Theorem 9 are satisfied and Equation (3) is solvable. Indeed, $x = 0.3 \leftarrow_P 0.5 = 0.6$ is a solution, as the next equality shows:

$$
(0.5 * 0.6) \vee (0.2 * n_P(0.6)) = (0.5 * 0.6) \vee (0.2 * 0) = 0.3 \vee 0 = 0.3
$$

Moreover, $x = 0.3 \leftarrow_P 0.5 = 0.6$ is the unique solution since $x = 0$ does not verify Equation (3), as we can see below:

$$
(0.5 * 0) \vee (0.2 * n_P(0)) = (0.5 * 0) \vee (0.2 * 1) = 0 \vee 0.2 = 0.2 \neq 0.3
$$

□

Once we have studied bipolar max-product fuzzy relation equations with the product negation and containing only one unknown variable, our following goal consists in increasing the number of unknown variables appearing in a bipolar max-product FRE with the product negation and studying sufficient and necessary conditions to guarantee its solvability.

3.2. Solving bipolar max-product FREs with different unknown variables

This section includes results associated with the solvability of bipolar max-product FREs with the product negation and a finite number of different unknown variables. Besides, we will present the conditions under which the existence of either a greatest (least, respectively) solution or a finite number of maximal (minimal, respectively) solutions can be guaranteed in these equations.

To begin with this section, we will present a result which characterizes the solvability of a bipolar max-product FRE when the independent term is equal to zero. The consideration of this case will allow us to obtain under what conditions a general bipolar max-product FRE with different unknown variables is solvable in an easier way.

Theorem 11. *Given $a_j^+, a_j^- \in [0, 1]$ and x_j an unknown variable belonging to $[0, 1]$, for all $j \in \{1, \ldots, m\}$. The bipolar max-product fuzzy relation equation*

$$\bigvee_{j=1}^{m} (a_j^+ * x_j) \vee (a_j^- * n_P(x_j)) = 0 \tag{4}$$

is solvable if and only if, for each $j \in \{1, \ldots, m\}$, either the equality $a_j^+ = 0$ or $a_j^- = 0$ is satisfied.

PROOF. First of all, we will suppose that, for each $j \in \{1, \ldots, m\}$, either the equality $a_j^+ = 0$ or $a_j^- = 0$ is satisfied, and we will find a solution of Equation (4). Given $k \in \{1, \ldots, m\}$, if $a_k^+ = 0$, then the variable $\hat{x}_k$ of the solution will be 1, since, in this case, we obtain:

$$(a_k^+ * \hat{x}_k) \vee (a_k^- * n_P(\hat{x}_k)) = 0 \vee (a_k^- * 0) = 0$$

Otherwise, if $a_k^- = 0$, then we consider $\hat{x}_k = 0$, which leads us to the same result:

$$(a_k^+ * \hat{x}_k) \vee (a_k^- * n_P(\hat{x}_k)) = (a_k^- * 0) \vee 0 = 0$$

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Therefore, the obtained tuple $(\hat{x}_1, \dots, \hat{x}_m)$ verifies that:

$$\bigvee_{j=1}^{m} (a_j^+ * \hat{x}_j) \vee (a_j^- * n_P(\hat{x}_j)) = 0$$

and, as a consequence, it is a solution of Equation (4).

Now, we suppose that Equation (4) is solvable and the tuple $(x_1, \dots, x_m)$ is a solution of such equation. We will assume that there exists $k \in \{1, \dots, m\}$ such that $a_k^+ > 0$ and $a_k^- > 0$, and we will obtain a contradiction.

If $x_k = 0$, then $n_P(x_k) = 1$, and therefore the chain of inequalities below holds

$$\bigvee_{j=1}^{m} (a_j^+ * x_j) \vee (a_j^- * n_P(x_j)) \geq (a_k^- * n_P(x_k)) = a_k^- > 0$$

which contradicts that $(x_1, \dots, x_m)$ is a solution of Equation (4). On the other hand, if $x_k > 0$, due to the fact that $a_k^+ > 0$, we obtain the same strict inequality:

$$\bigvee_{j=1}^{m} (a_j^+ * x_j) \vee (a_j^- * n_P(x_j)) \geq (a_k^+ * x_k) > 0$$

which also contradicts the hypothesis. Thus, we can ensure that either the equality $a_j^+ = 0$ or $a_j^- = 0$ holds, for all $j \in \{1, \dots, m\}$. $\qquad \square$

The following theorem characterizes the solvability of the proper bipolar max-product FREs containing different unknown variables.

Theorem 12. *Given $a_j^+, a_j^- \in [0, 1]$, $b \in (0, 1]$ and x_j an unknown variable belonging to $[0, 1]$, for all $j \in \{1, \dots, m\}$. The bipolar max-product fuzzy relation equation*

$$\bigvee_{j=1}^{m} (a_j^+ * x_j) \vee (a_j^- * n_P(x_j)) = b \qquad (5)$$

is solvable if and only if $b \leq \max\{a_j^+ \mid j \in \{1, \dots, m\}\}$ or there exists $k \in \{1, \dots, m\}$ such that $a_k^- = b$.

PROOF. We will suppose that $b \leq \max\{a_j^+ \mid j \in \{1, \dots, m\}\}$ and we will prove that Equation (5) is solvable. Clearly, $J = \{j \in \{1, \dots, m\} \mid b \leq a_j^+\}$ is a non-empty set. Furthermore, by definition of the product residuated implication, the following statements hold, for each $j \in \{1, \dots, m\}$:

(i) If $j \in J$, then $b \leftarrow_P a_j^+ = b/a_j^+$ [1] and we obtain the following equalities $a_j^+ * (b \leftarrow_P a_j^+) = b$ and $a_j^- * n_P(b \leftarrow_P a_j^+) = 0$.

(ii) If $j \notin J$, then we obtain that $b \leftarrow_P a_j^+ = 1$ and the following expressions hold $a_j^+ * (b \leftarrow_P a_j^+) = a_j^+ < b$ and $a_j^- * n_P(b \leftarrow_P a_j^+) = 0$.

From (i) and (ii), we can conclude that the tuple $(b \leftarrow_P a_1^+, \ldots, b \leftarrow_P a_m^+)$ is a solution of Equation (5).

Now, we will suppose that there exists $k \in \{1, \ldots, m\}$ such that $a_k^- = b$ and $\max\{a_j^+ \mid j \in \{1, \ldots, m\}\} < b$ and we will demonstrate that Equation (5) is solvable. Notice that, if $b \leq \max\{a_j^+ \mid j \in \{1, \ldots, m\}\}$, then we are in the case above, and thus Equation (5) is solvable. By hypothesis, $a_j^+ < b$ for each $j \in \{1, \ldots, m\}$, and therefore, $b \leftarrow_P a_j^+ = 1$. As a consequence, $a_j^+ * (b \leftarrow_P a_j^+) = a_j^+ < b$ and $a_j^- * n_P(b \leftarrow_P a_j^+) = 0$, for each $j \in \{1, \ldots, m\}$. Hence, we obtain straightforwardly that the tuple $(\hat{x}_1, \ldots, \hat{x}_k, \ldots, \hat{x}_m)$ with $\hat{x}_k = 0$ and $\hat{x}_j = 1$, for each $j \in \{1, \ldots, m\}$ such that $j \neq k$, is a solution of Equation (5).

The proof of the counterpart is analogous to the proof of Theorem 9. $\square$

From now on, explanatory examples will be included for a better understanding of each result.

Example 13. This example will consider the bipolar max-product FRE with three unknown variables $x_1, x_2, x_3 \in [0, 1]$ given by the following equation:

$$(0.4 * x_1) \vee (0.7 * n_P(x_1)) \vee (0.2 * x_2) \vee (0.1 * n_P(x_2)) \vee (0.5 * x_3) \vee (0.2 * n_P(x_3)) = 0.3 \quad (6)$$

We can ensure that Equation (6) is solvable because the hypothesis required in Theorem 12 are verified, that is, $\max\{0.4, 0.2, 0.5\} = 0.5 \geq 0.3$. Following an analogous reasoning to the one given in the proof of such theorem, we obtain that the tuple $(0.75, 1, 0.6)$ is a solution of Equation (6). In addition, we can make simple computations in order to obtain all possible solutions of Equation (6). For instance, the tuple $(0.5, 0, 0.6)$ is another solution of Equation (6):

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[1]Notice that, if $a_j^+ = 0$, then $a_j^+ < b$, and thus $a_j^+ \notin J$.

$$(0.4 * 0.5) \vee (0.7 * n_P(0.5)) \vee (0.2 * 0) \vee (0.1 * n_P(0)) \vee (0.5 * 0.6) \vee (0.2 * n_P(0.6)) \quad =$$
$$0.2 \vee 0 \vee 0 \vee 0.1 \vee 0.3 \vee 0 \quad =$$
$$0.3$$

$\square$

In view of the results obtained in the previous example, one can ask whether $(0.75, 1, 0.6)$ is the greatest solution of Equation (6) or if there exists a finite number of maximal solutions for such equation. A similar question with respect to the existence of its least solution or different minimal solutions can also arise. These issues will be analyzed in the following section.

3.2.1. Computing maximal and minimal solutions

Next, we are interested in knowing when a bipolar max-product FRE with the product negation and different variables has either a greatest (least, respectively) solution or a finite number of maximal (minimal, respectively) solutions. To reach this goal, we need to introduce the following results.

Theorem 14. *Given $a_j^+, a_j^- \in [0, 1]$, $b \in (0, 1]$ and x_j an unknown variable in $[0, 1]$, for each $j \in \{1, \dots, m\}$. If Equation (5) is a solvable bipolar max-product FRE, then the following statements hold:*

(1) If $b \leq \max\{a_j^+ \mid j \in \{1, \dots, m\}\}$, the set of solutions of Equation (5) has a greatest element.

(2) If $a_j^+ < b$ for each $j \in \{1, \dots, m\}$, then the set of maximal solutions of Equation (5) is finite. Moreover, the number of maximal solutions is:

$$card(\{k \in \{1, \dots, m\} \mid a_k^- = b\})$$

PROOF. In order to prove Statement (1), suppose that the inequality $b \leq \max\{a_j^+ \mid j \in \{1, \dots, m\}\}$ holds. By an analogous reasoning to the proof given in Theorem 12, we obtain that the tuple $(b \leftarrow_P a_1^+, \dots, b \leftarrow_P a_m^+)$ is a solution of Equation (5). In the following, we will prove that $(b \leftarrow_P a_1^+, \dots, b \leftarrow_P a_m^+)$ is the greatest solution of Equation (5), by reduction to the absurd.

Hence, we suppose that there exists a tuple $(x_1, \dots, x_m)$ being a solution of Equation (5) such that $(b \leftarrow_P a_1^+, \dots, b \leftarrow_P a_m^+) < (x_1, \dots, x_m)$. Then,

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there exists $k \in \{1, \ldots, m\}$ such that $x_k > b \leftarrow_P a_k^+$. According to the adjoint property, we can assert that $a_k^+ * x_k > b$. As a result, we obtain that

$$\bigvee_{j=1}^{m} (a_j^+ * x_j) \vee (a_j^- * n_P(x_j)) = \bigvee_{j=1}^{m} (a_j^+ * x_j) \geq a_k^+ * x_k > b$$

which lead us to conclude that $(x_1, \ldots, x_m)$ is not a solution of Equation (5), in contradiction with the hypothesis. Hence, we can ensure that $(b \leftarrow_P a_1^+, \ldots, b \leftarrow_P a_m^+)$ is the greatest solution of Equation (5). That is, Statement (1) holds.

Now, assume that $a_j^+ < b$ for each $j \in \{1, \ldots, m\}$, in order to demonstrate Statement (2). Given $A = \{k \in \{1, \ldots, m\} \mid a_k^- = b\}$ and B the set of maximal solutions of Equation (5), we consider the mapping $f \colon A \to B$ defined as $f(k) = (1, \ldots, 1, 0, 1, \ldots, 1)$, being the element 0 in the k-th position of the tuple. In order to prove that the cardinal of A coincides with the cardinal of B, we will demonstrate that f is a bijection between A and B.

First of all, let us see that f is well-defined. Consider a fixed $k \in A$. By an analogous reasoning to the proof of Theorem 12, we can ensure that the tuple $(1, \ldots, 1, 0, 1, \ldots, 1)$ is a solution of Equation (5), being the element 0 in the k-th position of the tuple. Suppose that there exists another solution $(\hat{x}_1, \ldots, \hat{x}_k, \ldots, \hat{x}_m)$ of Equation (5) such that $(1, \ldots, 1, 0, 1, \ldots, 1) < (\hat{x}_1, \ldots, \hat{x}_k, \ldots, \hat{x}_m)$. Therefore, $\hat{x}_k > 0$ and $\hat{x}_j = 1$, for each $j \in \{1, \ldots, m\}$ with $j \neq k$, in order to $(\hat{x}_1, \ldots, \hat{x}_k, \ldots, \hat{x}_m)$ be a greater solution. As a consequence, $n_P(\hat{x}_j) = 0$ for each $j \in \{1, \ldots, m\}$ and so,

$$\begin{aligned}
\bigvee_{j=1}^{m} (a_j^+ * x_j) \vee (a_j^- * n_P(x_j)) &= \bigvee_{j=1}^{m} (a_j^+ * x_j) \\
&= a_1^+ \vee \cdots \vee (a_k^+ * x_k) \vee \cdots \vee a_m^+ \\
&< b
\end{aligned}$$

since, by hypothesis, $a_j^+ < b$ for each $j \in \{1, \ldots, m\}$. Consequently, the tuple $(\hat{x}_1, \ldots, \hat{x}_k, \ldots, \hat{x}_m)$ is not a solution of Equation (5). Thus, $(1, \ldots, 1, 0, 1, \ldots, 1)$ is a maximal solution of Equation (5), which permits us to assert that f is well-defined. Moreover, given $k_1, k_2 \in A$ with $k_1 \neq k_2$, then $f(k_1)$ is clearly different than $f(k_2)$, since the element 0 is in the k_1-th position in $f(k_1)$ while the element 0 appears in the k_2-th position in $f(k_2)$. Therefore, f is an order-embedding mapping.

Now, we will prove that f is also onto. Let $(x_1, \ldots, x_m) \in B$. Clearly, as $a_j^+ < b$ for each $j \in \{1, \ldots, m\}$, we have that $a_j^+ * x_j < b$. Therefore,

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as the tuple $(x_1, \ldots, x_m)$ is a solution of Equation (5), we can ensure that there exists $k \in \{1, \ldots, m\}$ such that $a_k^- * n_P(x_k) = b$. According to the definition of the product negation, we obtain that $n_P(x_k)$ can only take the values 0 and 1. Since $b \neq 0$, we deduce that $n_P(x_k) = 1$, and as a consequence, $x_k = 0$. This fact allows us to ensure that $a_k^- = b$, that is, $k \in A$. Due to $(x_1, \ldots, x_m)$ is a maximal solution of Equation (5) and the inequality $a_j^+ * x_j < b$ holds, for each $j \in \{1, \ldots, m\}$, we can assert that $x_j = 1$ for each $j \in \{1, \ldots, m\}$ with $j \neq k$. Hence, we obtain that $f(k) = (1, \ldots, 1, 0, 1, \ldots, 1) = (x_1, \ldots, x_m)$, being the element 0 in the k-th position of the tuple.

Due to the mapping f is a bijection, we can conclude that the number of maximal solutions is $card(\{k \in \{1, \ldots, m\} \mid a_k^- = b\})$. Equivalently, Statement (2) is satisfied. □

As a consequence of Theorem 14, the next corollary arises.

Corollary 15. *Given $a_j^+, a_j^- \in [0, 1]$, $b \in (0, 1]$ and x_j an unknown variable in $[0, 1]$, for each $j \in \{1, \ldots, m\}$. Consider that Equation (5) is a solvable bipolar max-product FRE, then the following statements hold:*

(1) *If $\max\{a_j^+ \mid j \in \{1, \ldots, m\}\} \geq b$, then the greatest solution of Equation (5) is given by the tuple $(b \leftarrow_P a_1^+, \ldots, b \leftarrow_P a_m^+)$.*
(2) *If $a_j^+ < b$ for each $j \in \{1, \ldots, m\}$, then the set of maximal solutions of Equation (5) is given by:*

$$\{(1, \ldots, 1, x_k, 1, \ldots, 1) \mid x_k = 0 \text{ with } k \in K_P^-\}$$

where $K_P^- = \{k \in \{1, \ldots, m\} \mid a_k^- = b\}$

We also have to distinguish different cases to ensure the existence of the least solution and the set of minimal solutions of a solvable bipolar max-product FRE with different variables. It is important to mention that the set of minimal solutions of such equation can be empty, as the following theorem shows.

Theorem 16. *Given $a_j^+, a_j^- \in [0, 1]$, $b \in (0, 1]$ and x_j an unknown variable belonging to $[0, 1]$, for each $j \in \{1, \ldots, m\}$. Consider that Equation (5) is a solvable bipolar max-product FRE. Then, the following statements hold:*

(1) *If there exists $k \in \{1, \ldots, m\}$ such that $a_k^- = b$ and $a_j^- \leq b$, for each $j \in \{1, \ldots, m\}$, then the set of solutions of Equation (5) has a least element.*

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(2) *If there exist $k_1, k_2 \in \{1, \ldots, m\}$ such that $a_{k_1}^- = b$ and $a_{k_2}^- > b$, then the set of solutions of Equation (5) has no minimal elements.*

(3) *If $a_j^- \neq b$ for each $j \in \{1, \ldots, m\}$, then the set of minimal solutions of Equation (5) is finite. Moreover, the number of minimal solutions is:*

$$card(\{k \in \{1, \ldots, m\} \mid a_k^+ \geq b \text{ and } a_j^- < b \text{ for each } j \neq k\})$$

PROOF. First of all, to prove Statement (1), we will suppose that there exists $k \in \{1, \ldots, m\}$ such that $a_k^- = b$ and $a_j^- \leq b$, for each $j \in \{1, \ldots, m\}$. Then, the tuple $(0, \ldots, 0)$ is clearly a solution of Equation (5), since:

$$\bigvee_{j=1}^{m} (a_j^+ * 0) \vee (a_j^- * n_P(0)) = \bigvee_{j=1}^{m} (a_j^- * n_P(0)) = \bigvee_{j=1}^{m} a_j^- = b$$

Clearly, any other solution of Equation (5) is greater than $(0, \ldots, 0)$. As a consequence, $(0, \ldots, 0)$ is the least element of the set of solutions of Equation (5). That is, Statement (1) is satisfied.

Now, in order to demonstrate Statement (2), we suppose that there exist $k_1, k_2 \in \{1, \ldots, m\}$ such that $a_{k_1}^- = b$ and $a_{k_2}^- > b$. We will prove that the set of solutions of Equation (5) has no minimal elements by reduction to the absurd.

Let us assume that $(x_1, \ldots, x_m)$ is a minimal solution of Equation (5). First of all, we will see that $x_{k_1} = 0$ and $x_{k_2} > 0$. On the one hand, suppose that $x_{k_1} > 0$. Since equalities $a_{k_1}^+ * 0 = 0$ and $a_{k_1}^- * n_P(0) = a_{k_1}^- = b$ are straightforwardly satisfied, we obtain that the tuple $(x_1, \ldots, x_{k_1-1}, 0, x_{k_1+1}, \ldots, x_m)$ is another solution of Equation (5), which is strictly smaller than $(x_1, \ldots, x_m)$. Thus, we can ensure than $x_{k_1} = 0$. On the other hand, let us assume that $x_{k_2} = 0$. In this case, as $a_{k_2}^- > b$, the following inequality holds

$$\bigvee_{j=1}^{m} (a_j^+ * x_j) \vee (a_j^- * n_P(x_j)) \geq a_{k_2}^- * n_P(x_{k_2}) = a_{k_2}^- > b$$

As a result, the tuple $(x_1, \ldots, x_m)$ is not a solution of Equation (5). Consequently, we can ensure that $x_{k_2} > 0$. Therefore, we can assert that $x_{k_1} = 0$ and $x_{k_2} > 0$. In addition, notice that $a_{k_1}^- * n_P(x_{k_1}) = b$. Due to the fact that $(x_1, \ldots, x_m)$ is a solution of Equation (5), the inequality $a_{k_2}^+ * x_{k_2} \leq b$ is verified. Then, as the product t-norm is a monotonic operator, we obtain that inequality $a_{k_2}^+ * \frac{x_{k_2}}{2} \leq b$ also holds. Notice that, since $x_{k_2} > 0$, then $\frac{x_{k_2}}{2} < x_{k_2}$. Since $a_{k_1}^- * n_P(x_{k_1}) = b$, the

tuple $(x_1, \ldots, x_{k_2-1}, \frac{x_{k_2}}{2}, x_{k_2+1} \ldots, x_m)$ is a solution of Equation (5), which satisfies $(x_1, \ldots, x_{k_2-1}, \frac{x_{k_2}}{2}, x_{k_2+1} \ldots, x_m) < (x_1, \ldots, x_m)$. As a result, we conclude that $(x_1, \ldots, x_m)$ is not a minimal solution, in contradiction with the hypothesis, and thus we can assert that Statement (2) holds.

Finally, we will show that Statement (3) is verified. Given $A = \{k \in \{1, \ldots, m\} \mid a_k^+ \geq b$ and $a_j^- \leq b$ for each $j \neq k\}$ and B the set of minimal solutions of Equation (5), we will demonstrate that the mapping $f \colon A \to B$, which associates each $k \in A$ with a tuple $(0, \ldots, 0, b \leftarrow_P a_k^+, 0 \ldots, 0)$, being the element $b \leftarrow_P a_k^+$ in the k-th position of the tuple, is a bijection.

Now, we will prove that f is well-defined. Consider a fixed $k \in A$. Since $a_k^+ \geq b$ and $b \neq 0$, we obtain that $b \leftarrow_P a_k^+ = b/a_k^+ > 0$ and consequently, the following equalities $a_k^+ * (b \leftarrow_P a_k^+) = b$ and $a_k^- * n_P(b \leftarrow_P a_k^+) = 0$ are satisfied. In addition, since the inequality $a_j^- < b$ holds for each $j \in \{1, \ldots, m\}$ with $j \neq k$, we can ensure that the tuple $(0, \ldots, 0, b \leftarrow_P a_k^+, 0, \ldots, 0)$ is a solution of Equation (5). Suppose that there exists another different tuple $(x_1, \ldots, x_k, \ldots, x_m)$ being solution of Equation (5) and satisfying that $(x_1, \ldots, x_k, \ldots, x_m) < (0, \ldots, 0, b \leftarrow_P a_k^+, 0, \ldots, 0)$. Then, we have that $x_k < b \leftarrow_P a_k^+$ and $x_j = 0$, for each $j \in \{1, \ldots, m\}$ with $j \neq k$. Clearly, as the product t-norm is strictly order-preserving, the inequality $a_k^+ * x_k < b$ is also verified. Finally since, by hypothesis, $a_j^- \neq b$ for each $j \in \{1, \ldots, m\}$, we can conclude that $a_j^- * n_P(x_j) \neq b$ for each $j \in \{1, \ldots, m\}$, and thus, $(x_1, \ldots, x_m)$ is not a solution of Equation (5). Therefore, $(0, \ldots, 0, b \leftarrow_P a_k^+, 0, \ldots, 0)$ is a minimal solution of Equation (5) and so, we can ensure that $f(k) = (0, \ldots, 0, b \leftarrow_P a_k^+, 0, \ldots, 0) \in B$, that is, the mapping f is well-defined.

In the following, we will see that f is an order-embedding mapping. Given $k_1, k_2 \in A$ with $k_1 \neq k_2$, the tuples $(0, \ldots, 0, b \leftarrow_P a_{k_1}^+, 0, \ldots, 0)$ and $(0, \ldots, 0, b \leftarrow_P a_{k_2}^+, 0, \ldots, 0)$ are clearly different.

It remains to prove that f is an onto mapping. Given a minimal solution $(x_1, \ldots, x_m) \in B$, we will prove that there exists an element $k \in A$ such that $f(k) = (x_1, \ldots, x_m)$. Taking into account that Equation (5) is solvable and $a_j^- \neq b$ for each $j \in \{1, \ldots, m\}$, by Theorem 12, we can ensure that $\max\{a_j^+ \mid j \in \{1, \ldots, m\}\} \geq b$. In addition, due to $(x_1, \ldots, x_m)$ is a solution of Equation (5), following a similar reasoning to the proof of Theorem 12, we obtain that there exists $k \in \{1, \ldots, m\}$ such that $a_k^+ \geq b$ and $x_k = b \leftarrow_P a_k^+$.

Now, suppose that there exists $j \in \{1, \ldots, m\}$ with $j \neq k$ such that $a_j^- > b$ and we will obtain a contradiction. Due to $(x_1, \ldots, x_m)$ is a solution of Equation (5) and $a_j^- > b$, then we can guarantee that $x_j \neq$

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0. As product t-norm is a strictly order-preserving mapping, the tuple $(x_1, \ldots, x_k, \ldots, \frac{x_j}{2}, \ldots, x_m)$ is also a solution of Equation (5) verifying that $(x_1, \ldots, x_k, \ldots, \frac{x_j}{2}, \ldots, x_m) < (x_1, \ldots, x_m)$. We have supposed, without loss of generality, that $k < j$. This fact leads us to a contradiction since $(x_1, \ldots, x_m)$ is a minimal solution of Equation (5), by hypothesis. Hence, we can conclude that the inequality $a_j^- \leq b$ holds, for each $j \in \{1, \ldots, m\}$ with $j \neq k$. That is, $k \in A$.

Hereafter, we will see that $f(k) = (x_1, \ldots, x_m)$. We have already proved that there exists $k \in \{1, \ldots, m\}$ such that $x_k = b \leftarrow_P a_k^+$ and the inequality $a_j^- \leq b$ holds, for each $j \in \{1, \ldots, m\}$ with $j \neq k$. Suppose that there exists $j \in \{1, \ldots, m\}$ with $j \neq k$ such that $x_j > 0$ and we will see that this fact contradicts that $(x_1, \ldots, x_m)$ is a minimal solution. Since $a_j^- \leq b$, we obtain $a_j^+ * 0 = 0$ and $a_j^- * n_P(0) = a_j^- \leq b$. As a result, we can ensure that the tuple $(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_m)$ is also a solution of Equation (5) satisfying that $(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_m) < (x_1, \ldots, x_j, \ldots, x_m)$. This is a contradiction with respect to $(x_1, \ldots, x_m)$ is a minimal solution of Equation (5). Therefore, the equality $x_j = 0$ is verified, for each $j \in \{1, \ldots, m\}$ with $j \neq k$. Finally, we can conclude that $f(k) = (0, \ldots, 0, b \leftarrow_P a_k^+, 0, \ldots, 0) = (x_1, \ldots, x_m)$, and thus Statement (3) holds. □

The following corollary is straightforwardly obtained from Theorem 16.

Corollary 17. *Given $a_j^+, a_j^- \in [0,1]$, $b \in (0,1]$, x_j an unknown variable belonging to $[0,1]$, for each $j \in \{1, \ldots, m\}$, and a solvable bipolar max-product FRE as in Equation (5). Then, the following statements hold:*

(1) If there exists $k \in \{1, \ldots, m\}$ such that $a_k^- = b$ and $a_j^- \leq b$, for each $j \in \{1, \ldots, m\}$, then the least solution of Equation (5) is $(0, \ldots, 0)$.

(2) If $a_j^- \neq b$ for each $j \in \{1, \ldots, m\}$, then the set of minimal solutions of Equation (5) is given by:

$$\{(0, \ldots, 0, x_k, 0, \ldots, 0) \mid x_k = b \leftarrow_P a_k^+ \text{ with } k \in K_P^+\}$$

where $K_P^+ = \{k \in \{1, \ldots, m\} \mid a_k^+ \geq b \text{ and } a_j^- < b \text{ for each } j \neq k\}$.

Next example clarifies the previous results about the existence both the greatest/least solution and maximal/minimal solutions analyzing the bipolar max-product FRE given in Example 13.

Example 18. In Example 13 we showed that Equation (6):

$$(0.4 * x_1) \vee (0.7 * n_P(x_1)) \vee (0.2 * x_2) \vee (0.1 * n_P(x_2)) \vee (0.5 * x_3) \vee (0.2 * n_P(x_3)) = 0.3$$

is solvable. Now, by Theorems 14 and 16, we will study whether this equation has a greatest/least solution and maximal/minimal solutions. Taking into account that the hypothesis required in Statement (1) of Theorem 14 is verified, that is $\max\{0.4, 0.2, 0.5\} \geq 0.3$, we can guarantee that the greatest solution exists, which is $(0.75, 1, 0.6)$.

Applying Theorem 16 (since condition required in Statement (3) is verified) we obtain that this equation has only one minimal solution since:

$$card(\{k \in \{1, 2, 3\} \mid a_k^+ \geq b \text{ and } a_j^- < b \text{ for each } j \neq k\}) = card\{1\} = 1$$

Making the corresponding computations, we have that $(0.75, 0, 0)$ is a minimal solution. It is worth highlighting that the tuple $(0.75, 0, 0)$ is not the least solution of Equation (6). Indeed, we can easily see that the tuples $(x, 0, 0.6)$ with $0 < x < 0.75$ are solutions of Equation (6), being $(0.75, 0, 0)$ and $(x, 0, 0.6)$ incomparable tuples. Moreover, as we show below, the tuple $(0, 0, 0.6)$ is not a solution of Equation (6):

$$(0.4 * 0) \vee (0.7 * 1) \vee (0.2 * 0) \vee (0.1 * 1) \vee (0.5 * 0.6) \vee (0.2 * 0) = 0.7 \neq 0.3$$

Now, modifying the previous equation, we can consider Equation (7) and apply Theorems 12 and 16 in order to assert that Equation (7) is solvable and it does not have minimal solutions.

$$(0.4 * x_1) \vee (0.7 * n_P(x_1)) \vee (0.2 * x_2) \vee (0.1 * n_P(x_2)) \vee (0.5 * x_3) \vee (0.9 * n_P(x_3)) = 0.3 \quad (7)$$

This fact is due to the number of minimal solutions in Equation (7) is:

$$card(\{k \in \{1, 2, 3\} \mid a_k^+ \geq b \text{ and } a_j^- < b \text{ for each } j \neq k\}) = card\{\varnothing\} = 0$$

$\square$

Once the theoretical foundations of the solvability of bipolar max-product FREs have been presented, a toy example is shown below in order to illustrate how a bipolar max-product FRE can be used to represent a real-world situation. Specifically, the overheating of a motor will be modeled by means of a bipolar max-product FRE. Furthermore, given an overheating value, the bipolar FRE is used to determine what is the overheating due to and how a technician should perform in order to solve it.

Example 19. The suitable behaviour of a motor directly depends on maintaining its temperature under a certain threshold. To carry out this task, a group of experts stated that controlling the level of water, the level of oil and the radiator fan is crucial.

Consider the variables $x_1, x_2 \in [0, 1]$ which represent the truth value of *low water* and *low oil*, respectively, where 1 indicates empty water/oil container and 0 that the quantity of water/oil has exceeded the permitted limit. Let $x_3 \in \{0, 1\}$ be a variable such that $x_3 = 0$ corresponds to the radiator fan is working and $x_3 = 1$ to the radiator fan is stopped. The level of overheating of the motor is represented by a value $b \in [0, 1]$, where 0 indicates a correct temperature and 1 a critical level of overheating.

After a technical study, the experts reached the following conclusions on the performance of the motor:

- The motor overheating is directly proportional to the lack of water, with proportionality constant 0.4. Nevertheless, an excess of water turns out to be even more dangerous for the motor, since in such case it overheats at 0.7.

  Hence, the overheating being caused by the level of water can be modeled by means of the expression $(0.4 * x_1) \vee (0.7 * n_P(x_1))$. Observe that, if the water container is almost full but not exceeding the limits then the level of overheating is low, since $x_1 > 0$ and therefore $n_P(x_1) = 0$.

- Similarly, the motor also overheats directly proportional to the lack of oil, but in this case the proportionality constant is 0.2. Besides, if the oil exceeds the permitted maximum, it causes an overheating of 0.1. This behaviour can be interpreted by using the expression $(0.2 * x_2) \vee (0.1 * n_P(x_2))$.

- The radiator fan has a problem and it sometimes suddenly stops. When this happens, the motor overheats up to 0.5. Furthermore, the usual behaviour of the radiator makes that the motor overheats at 0.2. In this case, we obtain the expression $(0.5 * x_3) \vee (0.2 * n_P(x_3))$.

Now, considering that the level of water, the level of oil and the performance of the radiator fan are the only factors which affect to the ovearheating and basing on the experts knowledge, we can model the behaviour of the level of overheating of the motor through the following bipolar max-product fuzzy relation equation:

$$(0.4 * x_1) \vee (0.7 * n_P(x_1)) \vee (0.2 * x_2) \vee (0.1 * n_P(x_2)) \vee (0.5 * x_3) \vee (0.2 * n_P(x_3)) = b \quad (8)$$

This equation is a useful tool in order to know the reasons/causes of a given overheating. For example, if a technician observes that the motor presents an overheating of 0.3, in order to know the values of the other variables (causes) which imply this level of overheating, we need to solve Equation (8) when $b = 0.3$.

Notice that, the solvability of the obtained equation was already studied in Examples 13 and 18. Therefore, we obtain that Equation (8) is solvable, where $(0.75, 1, 0.6)$ is its greatest solution and it has only one minimal solution, the tuple $(0.75, 0, 0)$.

The most critical cases are then $(0.75, 1, 0.6)$ and $(0.75, 0, 0)$. First of all, since the tuple $(0.75, 0, 0)$ is the unique minimal solution of Equation (8), we deduce that there are no solutions such that $x_1 = 0$, and thus, the water container is not over the permitted limit. Secondly, we can assert that the level of oil is not giving rise to the overheating, as the variable x_2 can be either 0, 1, or any value among them. Indeed, a careful sight leads us to conclude that the level of oil can be ignored in this matter, since in the worst case it may result in an overheating of 0.2, but this overheating is already obtained due to the bad conditions of the radiator fan.

Last but not least, from the greatest solution of Equation (8), the tuple $(0.75, 1, 0.6)$, we deduce that the variable x_3 cannot be equal to 1. Hence, according to the fact that $x_3 \in \{0, 1\}$, we conclude that $x_3 = 0$, and thus the radiator fan is working properly.

Consequently, basing on the conclusions which we have obtained, we would suggest to the technician refilling the water container, but being careful so that it does not exceed the limits. $\qquad \square$

3.3. Solving bipolar max-product FRE systems

After characterizing the solvability of bipolar max-product FREs with different variables and the product negation, and providing information about the algebraic structure of the set of solutions, we will give a further step to our research. The following goal will be to solve bipolar max-product FRE systems.

According to the results presented until now, one can think that the conditions required to solve an arbitrary system of bipolar max-product FREs will be very similar to the ones given in Theorems 9 and 12. However, the

following theorem shows that this reasoning is not true.

Theorem 20. *Let $a_{ij}^+, a_{ij}^-, x_j \in [0,1]$ and $b_i \in (0,1]$, for each $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$. Then the bipolar max-product FRE system given by*

$$\bigvee_{j=1}^m (a_{ij}^+ * x_j) \vee (a_{ij}^- * n_P(x_j)) = b_i, \qquad i \in \{1,\ldots,n\} \qquad (9)$$

is solvable if and only if there exist two index sets $J^+, J^- \subseteq \{1,\ldots,m\}$ with $J^+ \cap J^- = \varnothing$ such that, one of the following statements is verified, for each $i \in \{1,\ldots,n\}$:

(1) *there exists $j \in J^+$ such that $a_{ij}^+ \geq b_i$ and $b_i \leftarrow_P a_{ij}^+ \leq b_h \leftarrow_P a_{hj}^+$, for each $h \in \{1,\ldots,n\}$.*

(2) *there exists $j \in J^-$ such that $a_{ij}^- = b_i$ and $a_{hj}^- \leq b_h$ for each $h \in \{1,\ldots,n\}$.*

PROOF. Suppose that there exist two index sets $J^+, J^- \subseteq \{1,\ldots,m\}$ with $J^+ \cap J^- = \varnothing$, such that each $i \in \{1,\ldots,n\}$ satisfies either Statement (1) or (2). Consider the set $J_s^- = \{j \in J^- \mid a_{hj}^- \leq b_h$, for each $h \in \{1,\ldots,n\}\}$ and the tuple $(\hat{x}_1,\ldots,\hat{x}_m)$ defined, for each $j \in \{1,\ldots,m\}$, as:

$$\hat{x}_j = \begin{cases} 0 & \text{if} \quad j \in J_s^- \\ \min\{b_k \leftarrow_P a_{kj}^+ \mid k \in \{1,\ldots,n\}\} & \text{otherwise} \end{cases}$$

Now, we will see that $(\hat{x}_1,\ldots,\hat{x}_m)$ is a solution of System (9). Given $i \in \{1,\ldots,n\}$, by hypothesis, one of the following statements is verified:

- there exists $j \in J^+$ such that $a_{ij}^+ \geq b_i$ and $b_i \leftarrow_P a_{ij}^+ \leq b_h \leftarrow_P a_{hj}^+$, for each $h \in \{1,\ldots,n\}$. In this case, $\hat{x}_j = b_i \leftarrow_P a_{ij}^+$ (note that $j \notin J_s^-$ because $j \in J^+$ and $J^+ \cap J^- = \varnothing$). By the definition of the product implication, we obtain that $a_{ij}^+ * \hat{x}_j = b_i$. Since by hypothesis $b_i > 0$, we can ensure that $\hat{x}_j > 0$, and thus $n_P(\hat{x}_j) = 0$ and $a_{ij}^- * n_P(\hat{x}_j) = 0$.

- there exists $j \in J^-$ such that $a_{ij}^- = b_i$ and $a_{hj}^- \leq b_h$ for each $h \in \{1,\ldots,n\}$. Notice that, it is straightforwardly obtained $j \in J_s^-$. Therefore, $\hat{x}_j = 0$, and clearly $a_{ij}^+ * \hat{x}_j = 0$ and $a_{ij}^- * n_P(\hat{x}_j) = a_{ij}^- = b_i$.

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Therefore, we can ensure that there exists $j \in \{1, \ldots, m\}$ verifying that $(a_{ij}^+ * \hat{x}_j) \vee (a_{ij}^- * n_P(\hat{x}_j)) = b_i$.

In the following, we will prove that the elements $\hat{x}_{j'}$ in $(\hat{x}_1, \ldots, \hat{x}_m)$ with $j' \in \{1, \ldots, m\}$ and $j' \neq j$ verify $(a_{ij'}^+ * \hat{x}_{j'}) \vee (a_{ij'}^- * n_P(\hat{x}_{j'})) \leq b_i$. We have that, for every $j' \in \{1, \ldots, m\}$, the following statements hold:

- If $j' \in J_s^-$ then $a_{hj'}^- \leq b_h$, for each $h \in \{1, \ldots, n\}$. In particular, $a_{ij'}^- \leq b_i$. Therefore, as $\hat{x}_{j'} = 0$, we obtain that

$$(a_{ij'}^+ * \hat{x}_{j'}) \vee (a_{ij'}^- * n_P(\hat{x}_{j'})) = 0 \vee a_{ij'}^- = a_{ij'}^- \leq b_i$$

- If $j' \notin J_s^-$, then $\hat{x}_{j'} = \min\{b_h \leftarrow_P a_{hj'}^+ \mid h \in \{1, \ldots, n\}\}$. On the one hand, as $b_h > 0$ for each $h \in \{1, \ldots, n\}$, then $b_h \leftarrow_P a_{hj'}^+ > 0$ for each $h \in \{1, \ldots, n\}$. As a consequence, we obtain that $\hat{x}_{j'} > 0$, and thus $n_P(\hat{x}_{j'}) = 0$. On the other hand, by definition of $\hat{x}_{j'}$, we can ensure that, in particular, $\hat{x}_{j'} \leq b_i \leftarrow_P a_{ij'}^+$. Therefore, applying the adjoint property, the next inequality $a_{ij'}^+ * \hat{x}_{j'} \leq b_i$ holds. As a result, the following inequality is verified:

$$(a_{ij'}^+ * \hat{x}_{j'}) \vee (a_{ij'}^- * n_P(\hat{x}_{j'})) = (a_{ij'}^+ * \hat{x}_{j'}) \leq b_i$$

Therefore,

$$\bigvee_{j=1}^m (a_{ij}^+ * \hat{x}_j) \vee (a_{ij}^- * n_P(\hat{x}_j)) = b_i$$

By an analogous reasoning for each $i \in \{1, \ldots, n\}$, we can conclude that $(\hat{x}_1, \ldots, \hat{x}_m)$ is a solution of System (9).

In order to prove the counterpart, suppose that System (9) is solvable, and let build two sets $J^+, J^- \subseteq \{1, \ldots, m\}$ with $J^+ \cap J^- = \varnothing$ such that, for each $i \in \{1, \ldots, n\}$, either Statement (1) or Statement (2) is verified.

Given a solution $(\hat{x}_1, \ldots, \hat{x}_m)$ of System (9) and two index sets J^+, J^- defined as follows:

$$\begin{aligned} J^+ &= \{j \in \{1, \ldots, m\} \mid \hat{x}_j > 0\} \\ J^- &= \{j \in \{1, \ldots, m\} \mid \hat{x}_j = 0\} \end{aligned}$$

Clearly, $J^+ \cap J^- = \varnothing$. Given $i \in \{1, \ldots, n\}$, as $(\hat{x}_1, \ldots, \hat{x}_m)$ is a solution of System (9), one of the following statements is verified:

(a) there exists $j \in \{1, \ldots, m\}$ such that $a_{ij}^+ * \hat{x}_j = b_i$.

(b) there exists $j \in \{1, \ldots, m\}$ such that $a_{ij}^- * n_P(\hat{x}_j) = b_i$.

On the one hand, if Statement (a) holds, as $b_i > 0$, we deduce that $\hat{x}_j > 0$, and thus $j \in J^+$. Moreover, since operator $*$ is monotonic and $a_{ij}^+ * 1 = a_{ij}^+$, the equality $a_{ij}^+ * \hat{x}_j = b_i$ implies that $a_{ij}^+ \geq b_i$. In fact, by a similar reasoning to the one given in Theorem 9, it can be proved that $\hat{x}_j = b_i \leftarrow_P a_{ij}^+$.

Finally, suppose that there exists $h \in \{1, \ldots, n\}$ such that $b_i \leftarrow_P a_{ij}^+ > b_h \leftarrow_P a_{hj}^+$, this is, $\hat{x}_j > b_h \leftarrow_P a_{hj}^+$. From the adjoint property, we obtain that $\hat{x}_j \leq b_h \leftarrow_P a_{hj}^+$ if and only if $a_{hj}^+ * \hat{x}_j \leq b_h$, Therefore, we can ensure that $a_{hj}^+ * \hat{x}_j > b_h$. This fact leads us to a contradiction, since $(\hat{x}_1, \ldots, \hat{x}_m)$ would not be a solution of equation h in System (9) and so, it would not be a solution of the mentioned system.

Hence, we can ensure that, if Statement (a) holds, then there exists $j \in J^+$ such that $a_{ij}^+ \geq b_i$ and $b_i \leftarrow_P a_{ij}^+ \leq b_h \leftarrow_P a_{hj}^+$, for each $h \in \{1, \ldots, n\}$. That is, Statement (1) holds.

On the other hand, suppose that Statement (b) is verified. Since $b_i > 0$, the equality $a_{ij}^- * n_P(\hat{x}_j) = b_i$ implies that $n_P(\hat{x}_j) > 0$, and thus $\hat{x}_j = 0$. That is, $j \in J^-$. Consequently, we obtain that $n_P(\hat{x}_j) = 1$, and straightforwardly $a_{ij}^- = b_i$.

To end, suppose that there exists $h \in \{1, \ldots, n\}$ such that $a_{hj}^- > b_h$. As a result, we obtain that $(\hat{x}_1, \ldots, \hat{x}_m)$ is not a solution of equation h in System (9) and thus it is not a solution of that system. That is, we obtain a contradiction.

Therefore, if Statement (b) holds, then there exists $j \in J^-$ such that $a_{ij}^- = b_i$ and $a_{hj}^- \leq b_h$ for each $h \in \{1, \ldots, n\}$. In other words, Statement (2) holds. $\square$

The following example clarifies the previous result.

Example 21. Given the following bipolar max-product FRE system with three equations and three unknown variables

$$(1 * x_1) \vee (0.7 * n_P(x_1)) \vee (0.2 * x_2) \vee (0.4 * n_P(x_2)) \vee (0.5 * x_3) \vee (0.7 * n_P(x_3)) = 0.7$$
$$(0.8 * x_1) \vee (0.1 * n_P(x_1)) \vee (0.8 * x_2) \vee (0.2 * n_P(x_2)) \vee (0.3 * x_3) \vee (0.6 * n_P(x_3)) = 0.6$$
$$(0.4 * x_1) \vee (0.2 * n_P(x_1)) \vee (0.3 * x_2) \vee (0.3 * n_P(x_2)) \vee (0.6 * x_3) \vee (0.2 * n_P(x_3)) = 0.3$$

We will consider the index sets $J^+ = \{2, 3\}$ and $J^- = \{1\}$. We will see that for each $i \in \{1, 2, 3\}$ the conditions required in the hypothesis of Theorem 20 are verified. First of all, notice that $J^+ \cap J^- = \varnothing$.

- Case $i = 1$ (first equation of the system): there exists $j = 1$ belonging to J^- satisfying that $a_{11}^- = b_1 = 0.7$, $a_{21}^- = 0.1 \leq 0.6 = b_2$ and $a_{31}^- = 0.2 \leq 0.3 = b_3$.

- Case $i = 2$ (second equation of the system): we obtain that the index $j = 2 \in J^+$ verifies that $0.8 = a_{22}^+ \geq b_2 = 0.6$. In addition, the inequalities $b_2 \leftarrow_P a_{22}^+ \leq b_3 \leftarrow_P a_{32}^+$ and $b_2 \leftarrow_P a_{22}^+ \leq b_1 \leftarrow_P a_{12}^+$ hold, as we see below:

$$0.6 \leftarrow_P 0.8 = 0.75 \leq 1 = 0.7 \leftarrow_P 0.2$$

$$0.6 \leftarrow_P 0.8 = 0.75 \leq 1 = 0.3 \leftarrow_P 0.3$$

- Case $i = 3$ (third equation of the system): we find the index $j = 3$ belonging to J^+ satisfying that $0.6 = a_{33}^+ \geq b_3 = 0.3$. Furthermore, the next inequalities are obtained:

$$b_3 \leftarrow_P a_{33}^+ = 0.3 \leftarrow_P 0.6 = 0.5 \leq 1 = 0.7 \leftarrow_P 0.5 = b_1 \leftarrow_P a_{13}^+$$

$$b_3 \leftarrow_P a_{33}^+ = 0.3 \leftarrow_P 0.6 = 0.5 \leq 1 = 0.6 \leftarrow_P 0.3 = b_2 \leftarrow_P a_{23}^+$$

Consequently, Theorem 20 can be applied which leads us to obtain that the given system is solvable. Indeed, from the computations above, one can easily check that the tuple $(0, 0.75, 0.5)$ is a solution of such system.

Observe that, the tuple $(0, 0.75, 0.5)$ is not the unique solution of the system. We can find some other different definitions of the index sets J^+ and J^- such that, for each $i \in \{1, 2, 3\}$ the hypothesis of Theorem 20 are satisfied. For instance, considering the index sets $J^+ = \varnothing$ and $J^- = \{1, 2, 3\}$, it is easy to deduce that the system is solvable and the tuple $(0, 0, 0)$ is a solution of the given system.

It is also important to mention that, the set J^+ does not need to be the complementary set of J^-. That is, the equality $J^+ \cup J^- = \{1, 2, 3\}$ is not required in order to guarantee the solvability of the system. For example, the index sets $J^+ = \{1\}$ and $J^- = \{3\}$ verify the hypothesis of Theorem 20 and clearly $J^+ \cup J^- = \{1, 3\} \neq \{1, 2, 3\}$. From the sets $J^+ = \{1\}$ and $J^- = \{3\}$, we obtain the solution $(0.75, 0, 0)$. $\square$

We have provided sufficient and necessary conditions which allow us to know when an arbitrary bipolar max-product FRE system with the residuated negation of the product t-norm is solvable. In the future, we will study the resolution of bipolar max-product FRE systems in which independent terms can take the value zero.

4. Conclusions and future work

A broad study on the resolution of bipolar max-product FREs has been carried out, considering the non-involutive negation operators defined from the implication associated with the product t-norm. Three different parts can be distinguished in our research according to the complexity of the considered bipolar equations. The first part introduces a characterization on the resolution of bipolar max-product FREs with only one unknown variable. The second part shows under what conditions a bipolar max-product fuzzy relation equation containing different variables is solvable. Moreover, interesting properties related to the algebraic structure of the set of solutions have been included. The third part considers bipolar max-product FRE systems and presents the requirements to guarantee when these systems are solvable. Notice that, the residuated negations related to the Gödel implication and the product implication coincide, and therefore, we can ensure that the solvability for bipolar max-product FREs with the Gödel negation has also been analyzed in this paper.

As future work, based on Theorem 20, we will study the introduction of an efficient algorithm, determining whether a system of bipolar max-product fuzzy relation equations is solvable and computing at least one solution. It will be also fundamental to investigate more properties in order to know the algebraic structure of the complete set of solutions corresponding to an arbitrary solvable system of bipolar max-product fuzzy relation equations. Moreover, we are interested in studying bipolar fuzzy relation equations based on other general operators, such as uninorms, u-norms, adjoint triples, etc.

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