Generalized quantifiers in formal concept analysis

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Abstract

Usually, datasets contain imprecise data (noise), which can produce unspected results on the considered mappings. For instance, this can happen with the infimum and supremum operators, since both operators are straightforwardly associated with the universal and existencial quantifiers, respectively. An interesting possibility, of decreasing the impact of this possible noise in the final results, is the consideration of generalized quantifiers.

This paper introduces four kind of generalized quantifiers based on adjoint triples, which generalize the current approaches to a more flexible framework. Different properties and characterizations are studied and they have been applied to formal concept analysis, presenting the conjunctive and implicative concept-forming operators in this outstanding theory.

Keywords: Concept lattice, fuzzy set, generalized quantifier, adjoint triple

1. Introduction

Universal and existential quantifiers are considered in many operators, such as in the composition of boolean matrices, definition of crisp sets, etc. In these cases an element is considered if all the properties are satisfied (with the universal quantifier) or if only one property is required (with the existential quantifier). Hence, if there is some noise in the datasets, one object can wrongly not be considered or be considered, respectively. The natural generalization of universal and existential quantifiers to fuzzy sets and fuzzy logic is given by the infimum and supremum operators [6, 23, 25, 26, 34], which also heritage this drawback. In order to solve the gap between both notions, generalized quantifiers [10, 12, 24, 27, 28, 42, 43] have been introduced. These new operators

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are intermediate quantifiers between the universal and existential quantifiers, which can model fuzzy notions such as "Most" or "Many", providing less strict quantifiers to the applications.

On the other hand, formal concept analysis (FCA) [30] is a mathematical framework, which has become an appealing research topic both from theoretical [2, 3, 39] and applicative perspectives [1, 31, 29, 35, 41, 44]. The main goal of this setting is to obtain information from relational datasets through different branches. Two of the most relevant are the selection and classification of the variables (attributes) more important in the problem to be modeled [5**?** , 9, 21, 22, 32], and the computation of relationships among the variables (attribute implications) [8, 20, 14, 36]. These applications are based on the concept-forming operators, which are the two fundamental extractors of information. Multi-adjoint concept lattices [37, 38, 40] arose as a flexible fuzzy FCA framework in which versatile algebraic structures can be considered [17, 22], which generalizes other approaches [4, 6, 11].

In this paper, four alternative definitions of generalized quantifiers will be defined in the general algebraic structure of multi-adjoint lattices. We will show that these definitions satisfy similar properties of the monadic quantifiers of type $\langle 1 \rangle$ introduced in [13, 43] (determined by fuzzy measures [27]), as well as the useful characterization to simplify the expression of the original definition. Continuing the first approximation given in [15], this paper will study the use of these new quantifiers on the definitions of the concept-forming operators in order to decrease the universal behaviour of the infimum in these operators. We will focus the attention on the conjunctive and implicative definitions, showing the main properties, a useful characterization and the relationship with other interesting approaches. Specifically, we will prove that the implicative concept-forming operators generalize the ones defined in threshold concept lattices [7, 19, 45], which shows the narrow relationship between these operators and the implicative quantifiers. Thus, the new generalized quantifiers based on adjoint triples offer more flexible quantifiers, and their consideration in FCA provides alternative concept-forming operators that can better absorb some possible noise usually presents in datasets.

The structure of the paper is as follows. Section 2 recalls the main needed notions of multi-adjoint concept lattices. The conjunctive and implicative generalized quantifiers are introduced in Section 3, which are used in Section 4 to define the conjunctive and implicative quantifiers concept-forming operators. This section also contains the characterizations, the relationship with the threshold concept lattices and different properties. The paper ends with the conclusions and prospects for future work.

2. Multi-adjoint concept lattices

Multi-adjoint concept lattices arose as a new fuzzy approach to Formal Concept Analysis [40]. Specifically, the philosophy of the multi-adjoint paradigm was applied to obtain a general framework that could suitably accommodate different fuzzy approaches given in the literature [6, 11, 33]. Before including the main notions of the multi-adjoint concept lattices framework, we need to recall the basic operators to carry out the calculus in this framework.

Definition 1. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be mappings. We say that $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ if the following double equivalence is satisfied:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

for all $x \in P_1$, $y \in P_2$ and $z \in P_3$. The previous double equivalence is called *adjoint property.*

Adjoint triples are an interesting generalization of triangular norms and their residuated implications, since they preserve their main properties but the conjunctors are required to be neither commutative nor associative. A detailed study of adjoint triples was presented in [16, 18], where the next properties were proven.

Proposition 2. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$, then the following properties are satisfied:*

1. *$\&$ is order-preserving on both arguments.*
2. *$\swarrow$ and $\nwarrow$ are order-preserving on the first argument and order-reversing on the second argument.*
3. *$\bot_1 \,\&\, y = \bot_3$, $\top_3 \swarrow y = \top_1$, for all $y \in P_2$, when $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_3, \leq_3, \bot_3, \top_3)$ are bounded posets.*
4. *$x \,\&\, \bot_2 = \bot_3$ and $\top_3 \nwarrow x = \top_2$, for all $x \in P_1$, when $(P_2, \leq_2, \bot_2, \top_2)$ and $(P_3, \leq_3, \bot_3, \top_3)$ are bounded posets.*
5. *$z \nwarrow \bot_1 = \top_2$ and $z \swarrow \bot_2 = \top_1$, for all $z \in P_3$, when $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_2, \leq_2, \bot_2, \top_2)$ are bounded posets.*

After introducing the definition of adjoint triple and some properties that will be used later, we are in a position to recall the notions of multi-adjoint frame and context.

Definition 3.

- A *multi-adjoint frame* is a tuple $(L_1, L_2, P, \&_1, \dots, \&_n)$ where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P, for all $i \in \{1, \dots, n\}$.

- A *context* is a tuple (A, B, R, σ) such that A and B are non-empty sets, R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon A \times B \to \{1, \dots, n\}$ is a mapping which associates any element in $A \times B$ with some particular adjoint triple in the multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$.

Given a multi-adjoint frame and a context for that frame, different pieces of information from databases containing a set of attributes A and a set of objects B, related to each other by a binary relation $R \subseteq A \times B$, can be identified. These pieces of information are called multi-adjoint concepts and a hierarchy can be established on them providing an algebraic structure called multi-adjoint concept lattice.

Definition 4. Let $(L_1, L_2, P, \&_1, \ldots, \&_n)$ be a multi-adjoint frame and (A, B, R, σ) be a context.

- The *concept-forming operators* are the mappings ${}^{\uparrow}\colon L_2^B \longrightarrow L_1^A$ and ${}^{\downarrow}\colon L_1^A \longrightarrow L_2^B$ defined, for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, as:

$$\begin{aligned} g^{\uparrow}(a) &= \bigwedge_{b \in B} \Big(R(a,b) \swarrow^{\sigma(a,b)} g(b) \Big) \\ f^{\downarrow}(b) &= \bigwedge_{a \in A} \big(R(a,b) \nwarrow_{\sigma(a,b)} f(a) \big) \end{aligned}$$

- A *multi-adjoint concept* is a pair $\langle g, f \rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^{\uparrow} = f$ and $f^{\downarrow} = g$.

- A *multi-adjoint concept lattice* is the set

$$\mathcal{M} = \{ \langle g, f \rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow} = f, f^{\downarrow} = g \}$$

  in which the ordering is defined by $\langle g_1, f_1 \rangle \preceq \langle g_2, f_2 \rangle$ if and only if $g_1 \preceq_2 g_2$, or equivalently $f_2 \preceq_1 f_1$.

3. Conjunctive and implicative generalized quantifiers

Generalized quantifiers were introduced by Štěpnička and Holčapek in [43] in order to increase the versatility of fuzzy relational compositions in applications. We are interested in generalizing the notion of generalized quantifier and in order to reach this goal, two different approaches for defining generalized quantifiers, by using either the conjunctor or the implications involved in adjoint triples, will be presented. These approaches will increase the flexibility of the notion of generalized quantifier given in [43].

Continuing on the line proposed by Štěpnička and Holčapek [43], we will use fuzzy measures invariant with respect to the cardinality to define new versions of generalized quantifier. For that reason, we need to recall the notion of fuzzy measure invariant with respect to the cardinality.

Definition 5. Let $\mathcal{U}$ be a finite universe and $\mathcal{P}(\mathcal{U})$ be the powerset of $\mathcal{U}$. We will say that the mapping $\mu\colon \mathcal{P}(\mathcal{U}) \to [0,1]$ is a *fuzzy measure*, if it is an increasing mapping satisfying that $\mu(\varnothing) = 0$ and $\mu(\mathcal{U}) = 1$. We will say that the fuzzy measure μ *is invariant with respect to the cardinality*, if the following condition holds:

$$\text{If } |A| = |B| \text{ then } \mu(A) = \mu(B), \text{ for all } A, B \in \mathcal{P}(\mathcal{U})$$

where $|\cdot|$ denotes the cardinality of a set.

To illustrate the previous notion, we will show two fuzzy measures which were exemplified in [43]. Specifically, the mapping $\mu_{rc}\colon \mathcal{P}(\mathcal{U}) \to [0,1]$ defined as $\mu_{rc}(A) = \frac{|A|}{|\mathcal{U}|}$, for all $A \in \mathcal{P}(\mathcal{U})$, is a fuzzy measure invariant with respect to the cardinality. Notice that, if $\varphi\colon [0,1] \to [0,1]$ is an increasing mapping such that $\varphi(0) = 0$ and $\varphi(1) = 1$, then the mapping $\mu_\varphi\colon \mathcal{P}(\mathcal{U}) \to [0,1]$ defined as $\mu_\varphi(A) = \varphi(\mu_{rc}(A))$, for all $A \in \mathcal{P}(\mathcal{U})$, is also a fuzzy measure invariant with respect to the cardinality. The former fuzzy measure μ_{rc} is called *relative cardinality* whereas the latter fuzzy measure μ_φ is called *relative cardinality modified by* φ or simply *modified relative cardinality.*

After recalling and exemplifying the concept of fuzzy measure invariant with respect to the cardinality, we proceed to expose in detail the developed research work.

3.1. Conjunctive generalized quantifiers

This section will provide a new definition of generalized quantifier where the calculations with the fuzzy measure are computed by using the conjunctor of an adjoint triple. Properties related to different computation procedures for computing the proposed of generalized quantifier in an efficient way will be presented. It is convenient to mention that the proofs of these properties will clarify different parts of the ones given in [43]. From now on, we will consider adjoint triples defined on the complete lattice $([0,1], \leq)$ assuming that their conjunctors have 1 as left and right identity element.

Definition 6. Let $\mathcal{U}$ be a non-empty finite universe, $\mathcal{F}(\mathcal{U}) = [0,1]^{\mathcal{U}}$ be the set of fuzzy sets of $\mathcal{U}$ on $[0,1]$, $\mathcal{P}(\mathcal{U})$ be the powerset of $\mathcal{U}$, $\mu\colon \mathcal{P}(\mathcal{U}) \to [0,1]$ be a fuzzy measure invariant with respect to the cardinality and $(\&, \swarrow, \nwarrow)$ be an adjoint triple w.r.t $([0,1], \leq)$ such that $x \,\&\, 1 = 1 \,\&\, x = x$, for all $x \in [0,1]$.

- A mapping $Q_\mu\colon \mathcal{F}(\mathcal{U}) \to [0,1]$ defined, for all $C \in \mathcal{F}(\mathcal{U})$, as:

$$Q_\mu(C) = \bigvee_{D \in \mathcal{P}(\mathcal{U})} \left(\left(\bigwedge_{u \in D} C(u) \right) \& \ \mu(D) \right) \tag{1}$$

  is called *right conjunctive generalized quantifier determined by the fuzzy measure μ.*

- A mapping ${}_\mu Q\colon \mathcal{F}(\mathcal{U}) \to [0,1]$ defined, for all $C \in \mathcal{F}(\mathcal{U})$, as:

$${}_\mu Q(C) = \bigvee_{D \in \mathcal{P}(\mathcal{U})} \left(\mu(D) \ \& \left(\bigwedge_{u \in D} C(u) \right) \right) \tag{2}$$

  is called *left conjunctive generalized quantifier determined by the fuzzy measure μ.*

Notice that, in order to present aesthetic formulas above, we have considered that D be the empty set although this case does not affect to the computation since it provides the null value 0.

The universal and existential quantifiers can be obtained from Definition 6 considering the minimum and maximum fuzzy measures, respectively, as the following proposition shows. An analogous result can be obtained for a left conjunctive generalized quantifier.

Proposition 7. *Given a non-empty finite universe $\mathcal{U}$, the right conjunctive generalized quantifiers $Q_\forall$ and $Q_\exists$ determined by the minimum and maximum fuzzy measures $\mu_\forall$ and $\mu_\exists$, respectively, which are defined as follows:*

$$\mu_\forall(D) = \begin{cases} 1 & \textit{if } D = \mathcal{U} \\ 0 & \textit{otherwise} \end{cases} \qquad \mu_\exists(D) = \begin{cases} 0 & \textit{if } D = \varnothing \\ 1 & \textit{otherwise} \end{cases}$$

represent the universal and existencial quantifiers. That is, for all $C \in \mathcal{F}(\mathcal{U})$, the following equalities are satisfied:

$$\begin{aligned} Q_\forall(C) &= \bigwedge_{u \in \mathcal{U}} C(u) \\ Q_\exists(C) &= \bigvee_{u \in \mathcal{U}} C(u) \end{aligned}$$

Proof. Considering the universal quantifier $Q_\forall$, we have that the following chain of equalities hold, for all $C \in \mathcal{F}(\mathcal{U})$.

$$\begin{aligned} Q_\forall(C) &= \bigvee_{\substack{D \in \mathcal{P}(\mathcal{U}) \\ D \neq \mathcal{U}}} \left(\left(\bigwedge_{u \in D} C(u) \right) \mathbin{\&} \mu_\forall(D) \right) \vee \left(\left(\bigwedge_{u \in \mathcal{U}} C(u) \right) \mathbin{\&} \mu_\forall(\mathcal{U}) \right) \\ &= \bigvee_{\substack{D \in \mathcal{P}(\mathcal{U}) \\ D \neq \mathcal{U}}} \left(\left(\bigwedge_{u \in D} C(u) \right) \mathbin{\&} 0 \right) \vee \left(\left(\bigwedge_{u \in \mathcal{U}} C(u) \right) \mathbin{\&} 1 \right) \\ &\overset{(a)}{=} \bigwedge_{u \in \mathcal{U}} C(u) \end{aligned}$$

where (a) is obtained by Property 4 of Proposition 2 and taking into account that & has 1 as right identity element.
Now, we will carry out the proof for the existencial quantifier. For all $C \in \mathcal{F}(\mathcal{U})$, we obtain that:

$$
\begin{aligned}
Q_{\exists}(C) &= \bigvee_{D \in \mathcal{P}(\mathcal{U})} \left(\left(\bigwedge_{u \in D} C(u) \right) \,\&\, \mu_{\exists}(D) \right) \\
&= \bigvee_{D \in \mathcal{P}(\mathcal{U})} \left(\left(\bigwedge_{u \in D} C(u) \right) \,\&\, 1 \right) \\
&\overset{(b)}{=} \bigvee_{D \in \mathcal{P}(\mathcal{U})} \left(\bigwedge_{u \in D} C(u) \right) \\
&\overset{(c)}{=} \bigvee_{u \in \mathcal{U}} C(u)
\end{aligned}
$$

where (b) holds because $\&$ has 1 as right identity element and (c) is obtained since, there exists $u_{\vee} \in \mathcal{U}$, such that $C(u_{\vee}) = \bigvee_{u \in \mathcal{U}} C(u)$ and clearly the supremum is reachable for $D = \{u_{\vee}\}$. $\square$

It is important to note that the notion of right/left conjunctive generalized quantifier given in Definition 6 is not suitable, from a computational point of view, since the computation over all sets from $\mathcal{P}(\mathcal{U})$ is required. We propose an alternative computation procedure which makes use of the property of being invariant with respect to the cardinality of fuzzy measures, in order to compute the conjunctive generalized quantifier in a more efficient way.

Theorem 8. *Let $\mathcal{U} = \{u_1, \dots, u_n\}$ be a universe and Q_μ and ${}_\mu Q$ be the right and left conjunctive generalized quantifiers, respectively, determined by a fuzzy measure μ invariant with respect to the cardinality. Then,*

$$
\begin{aligned}
Q_\mu(C) &= \bigvee_{i=1}^{n} C(u_{\pi(i)}) \,\&\, \mu(\{u_1, \dots, u_i\}) \\
{}_\mu Q(C) &= \bigvee_{i=1}^{n} \mu(\{u_1, \dots, u_i\}) \,\&\, C(u_{\pi(i)})
\end{aligned}
$$

for all $C \in \mathcal{F}(\mathcal{U})$ and where π is a permutation on $\{1, 2, \dots, n\}$ such that $C(u_{\pi(1)}) \geq C(u_{\pi(2)}) \geq \dots \geq C(u_{\pi(n)})$.

Proof. Since both equalities are similar, only the first one will be proved. Consider an arbitrary fuzzy set $C \in \mathcal{F}(\mathcal{U})$ and a permutation π on $\{1, 2, \dots, n\}$, such that $C(u_{\pi(1)}) \geq C(u_{\pi(2)}) \geq \dots \geq C(u_{\pi(n)})$. Hence, we can ensure that the following equality holds:

$$
C(u_{\pi(i)}) = \bigwedge_{u \in \{u_{\pi(1)}, \dots, u_{\pi(i)}\}} C(u)
$$

which implies that, if $|D| = i$ then:

$$C(u_{\pi(i)}) \ \&\ \mu(\{u_{\pi(1)}, ..., u_{\pi(i)}\}) = \left(\bigwedge_{u \in \{u_{\pi(1)}, ..., u_{\pi(i)}\}} C(u) \right) \&\ \mu(D) \tag{3}$$

Moreover, if $|D| = i$ then there exists $j \geq i$ such that $u_{\pi(j)} \in D$ and therefore, the following chain of inequalities is satisfied:

$$\bigwedge_{u \in D} C(u) \leq C(u_{\pi(j)}) \leq C(u_{\pi(i)}) = \bigwedge_{u \in \{u_{\pi(1)}, ..., u_{\pi(i)}\}} C(u)$$

Therefore, by the monotonicity of &, we obtain that:

$$\left(\bigwedge_{u \in D} C(u) \right) \&\ \mu(D) \leq \left(\bigwedge_{u \in \{u_{\pi(1)}, ..., u_{\pi(i)}\}} C(u) \right) \&\ \mu(D) \tag{4}$$

Moreover, by the supremum property, we have that:

$$\begin{aligned} \bigvee_{\substack{D \in \mathcal{P}(\mathcal{U}) \\ |D| = i}} \left(\bigwedge_{u \in D} C(u) \right) \& \mu(D) = & \bigvee_{\substack{D \in \mathcal{P}(\mathcal{U}) \\ |D| = i \\ D \neq \{u_{\pi(1)}, ..., u_{\pi(i)}\}}} \left(\bigwedge_{u \in D} C(u) \right) \& \mu(D) \\ & \bigvee \left(\bigwedge_{u \in \{u_{\pi(1)}, ..., u_{\pi(i)}\}} C(u) \right) \& \mu(D) \\ & \stackrel{(a)}{=} \left(\bigwedge_{u \in \{u_{\pi(1)}, ..., u_{\pi(i)}\}} C(u) \right) \& \mu(D) \\ & \stackrel{(b)}{=} C(u_{\pi(i)}) \ \&\ \mu(\{u_1, ..., u_i\}) \end{aligned} \tag{5}$$

where (a) is deduced by Equation (4) and (b) by Equation (3). Finally, the following chain of equalities can be deduced:

$$\begin{aligned} Q_\mu(C) &= \bigvee_{D \in \mathcal{P}(\mathcal{U})} \left(\left(\bigwedge_{u \in D} C(u) \right) \&\ \mu(D) \right) \\ &= \bigvee_{i=1}^{n} \left(\bigvee_{\substack{D \in \mathcal{P}(\mathcal{U}) \\ |D| = i}} \left(\left(\bigwedge_{u \in D} C(u) \right) \&\ \mu(D) \right) \right) \\ &\stackrel{(c)}{=} \bigvee_{i=1}^{n} C(u_{\pi(i)}) \ \&\ \mu(\{u_1, ..., u_i\}) \end{aligned}$$

where (c) holds from Equation (5). $\square$

The following result proposes to compute the right and left conjunctive generalized quantifiers considering a fuzzy measure built from the relative cardinality, which arises as a direct consequence of Theorem 3.

Corollary 9. *Let $\mathcal{U} = \{u_1, \dots, u_n\}$ be a universe, $\varphi\colon \{1, \dots, n\} \to [0, 1]$ be an increasing mapping such that $\varphi(1) = 1$ and μ be a fuzzy measure built from the relative cardinality, by using φ. Then, for all $C \in \mathcal{F}(\mathcal{U})$, we have that*

$$\begin{aligned} Q_\mu(C) &= \bigvee_{i=1}^{n} C(u_{\pi(i)}) \,\&\, \varphi(i/n) \\ {}_\mu Q(C) &= \bigvee_{i=1}^{n} \varphi(i/n) \,\&\, C(u_{\pi(i)}) \end{aligned}$$

where π is a permutation on $\{1, 2, \dots, n\}$, such that $C(u_{\pi(1)}) \geq C(u_{\pi(2)}) \geq \dots \geq C(u_{\pi(n)})$.

3.2. Implicative generalized quantifiers

In this section we are going to extend the notion of generalized quantifier to the case in which the measure is operated through the implication $\nwarrow$ or $\swarrow$. We consider the same framework of the previous section, i.e., we will work with an adjoint triple $(\&, \nwarrow, \swarrow)$ defined in the complete lattice $([0, 1], \leq)$ and we will require $\&$ to satisfy the boundary condition with 1 in both arguments. Likewise, the results analogous to those of the previous section are established. Its proofs are completely similar to those of the previous results, so they are omitted.

Definition 10. Let $\mathcal{U}$ be a non-empty finite universe, $\mathcal{F}(\mathcal{U}) = [0, 1]^{\mathcal{U}}$ be the set of fuzzy sets of $\mathcal{U}$ on $[0, 1]$, $\mathcal{P}(\mathcal{U})$ be the powerset of $\mathcal{U}$, $\mu\colon \mathcal{P}(\mathcal{U}) \to [0, 1]$ be a fuzzy measure and $(\&, \swarrow, \nwarrow)$ be an adjoint triple w.r.t $([0, 1], \leq)$ such that $x \,\&\, 1 = 1 \,\&\, x = x$, for all $x \in [0, 1]$.

- A mapping $Q_\mu^{\swarrow}\colon \mathcal{F}(\mathcal{U}) \to [0, 1]$ defined, for all $C \in \mathcal{F}(\mathcal{U})$, as:

$$Q_\mu^{\swarrow}(C) = \bigwedge_{D \in \mathcal{P}(\mathcal{U})} \left(\left(\bigvee_{u \in D} C(u) \right) \swarrow \mu_(D) \right) \tag{6}$$

  is called *down implicative generalized quantifier determined by the fuzzy measure μ.*

- A mapping $Q_\mu^{\nwarrow}\colon \mathcal{F}(\mathcal{U}) \to [0, 1]$ defined, for all $C \in \mathcal{F}(\mathcal{U})$, as:

$$Q_\mu^{\nwarrow}(C) = \bigwedge_{D \in \mathcal{P}(\mathcal{U})} \left(\left(\bigvee_{u \in D} C(u) \right) \nwarrow \mu(D) \right) \tag{7}$$

  is called *up implicative generalized quantifier determined by the fuzzy measure μ.*

In this case, the existential and universal quantifiers can be obtained from Definition 10 considering the minimum and maximum fuzzy measures, respectively. An analogous result can be obtained for the respective up implicative generalized quantifier.

Proposition 11. *Given a non-empty finite universe $\mathcal{U}$, the down implicative generalized quantifiers $Q^{\swarrow}_{\mu_\forall}$ and $Q^{\swarrow}_{\mu_\exists}$ determined by the minimum and maximum fuzzy measures $\mu_\forall$ and $\mu_\exists$ represent, respectively, the existencial and universal quantifiers. That is, for all $C \in \mathcal{F}(\mathcal{U})$, the following equalities are satisfied:*

$$\begin{aligned} Q^{\swarrow}_{\mu_\forall}(C) &= \bigvee_{u\in\mathcal{U}} C(u) \\ Q^{\swarrow}_{\mu_\exists}(C) &= \bigwedge_{u\in\mathcal{U}} C(u) \end{aligned}$$

When the measure is invariant with respect to the cardinality, the following characterizations hold, which are similar to the ones given to conjunctive generalized quantifier.

Theorem 12. *Let $\mathcal{U} = \{u_1, \ldots, u_n\}$ be a universe and $Q^{\swarrow}_{\mu}$ and $Q^{\nwarrow}_{\mu}$ be the down and up implicative generalized quantifiers, respectively, determined by a fuzzy measure μ invariant with respect to the cardinality. Then,*

$$Q^{\swarrow}_{\mu}(C) = \bigwedge_{i=1}^{n} C(u_{\pi(i)}) \swarrow \mu(\{u_1, ..., u_i\})$$
$$Q^{\nwarrow}_{\mu}(C) = \bigwedge_{i=1}^{n} C(u_{\pi(i)}) \nwarrow \mu(\{u_1, ..., u_i\})$$

for all $C \in \mathcal{F}(\mathcal{U})$ and where π is a permutation on $\{1, 2, \ldots, n\}$ such that $C(u_{\pi(1)}) \leq C(u_{\pi(2)}) \leq \cdots \leq C(u_{\pi(n)})$.

This last result can also be rewritten as follows.

Corollary 13. *Let $\mathcal{U} = \{u_1, \ldots, u_n\}$ be a universe, $\varphi\colon \{1, \ldots, n\} \to [0, 1]$ be an increasing mapping such that $\varphi(1) = 1$ and μ be a fuzzy measure built from the relative cardinality, by using φ. Then, for all $C \in \mathcal{F}(\mathcal{U})$, we have that*

$$\begin{aligned} Q^{\swarrow}_{\mu}(C) &= \bigwedge_{i=1}^{n} C(u_{\pi(i)}) \swarrow \varphi(i/n) \\ Q^{\nwarrow}_{\mu}(C) &= \bigwedge_{i=1}^{n} C(u_{\pi(i)}) \nwarrow \varphi(i/n) \end{aligned}$$

where π is a permutation on $\{1, 2, \ldots, n\}$, such that $C(u_{\pi(1)}) \leq C(u_{\pi(2)}) \leq \cdots \leq C(u_{\pi(n)})$.

4. Formal-concept operators with generalized quantifiers

This section will use generalized quantifiers to provide formal-concept operators with a weaker existential character. From the definitions introduced in the previous section we have two possibilities depending on if conjunctive or implicative generalized quantifiers are considered. From now on, a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$, with $L_1 = L_2 = P = [0,1]$, a context (A, B, R, σ) and an extra adjoint triple $(\&, \swarrow, \nwarrow)$ on $[0,1]$ will be fixed.

The following definition uses the conjunctive version to present a generalization of the usual fuzzy concept-forming operators on multi-adjoint concept lattices. Notice that four possibilities exist considering the two quantifiers Q_μ and ${}_\mu\mathrm{Q}$. Since all of them have similar main properties, only one will be defined next.

Definition 14. Given two families of fuzzy measures $\{\mu_A^b \mid b \in B\}$, $\{\mu_B^a \mid a \in A\}$ on A and B, respectively, which are invariant with respect to the cardinality and ${}_A^b\mathrm{Q}$, Q_B^a the quantifiers determined by the fuzzy measures μ_A^b and μ_B^a, respectively, the *conjunctive quantified concept-forming operators* are denoted as ${}^{\uparrow_A\mathrm{Q}} : L_2^B \longrightarrow L_1^A$ and ${}^{\downarrow^{\mathrm{Q}_B}} : L_1^A \longrightarrow L_2^B$, where L_2^B and L_1^A denote the set of fuzzy subsets $g\colon B \to L_2$ and $f\colon A \to L_1$, respectively, and are defined, for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, as:

$$g^{\uparrow_{\mathrm{Q}_B}}(a) \quad = \quad \bigvee_{X \in \mathcal{P}(B)} \left(\bigwedge_{b \in X} R(a,b) \swarrow^{\sigma(a,b)} g(b) \right) \& \, \mu_B^a(X) \tag{8}$$

$$f^{\downarrow^{A\mathrm{Q}}}(b) \quad = \quad \bigvee_{Y \in \mathcal{P}(A)} \mu_A^b(Y) \, \& \left(\bigwedge_{a \in Y} R(a,b) \nwarrow_{\sigma(a,b)} f(a) \right) \tag{9}$$

The previous definition has been selected from the four possibilities for the simple reason that $\&$ be analogous to the adjoint triples of the lattice, or even equal to one of them, evaluating "attributes" in the left argument and "objects" in the right one.

From Corollary 9, we obtain the following characterization of the conjunctive quantified concept-forming operators, where the fuzzy sets φ_A^b and φ_B^a are defined from the two fuzzy measures μ_A^b and μ_B^a as follows:

$$\begin{aligned} \varphi_A^b(j/m) &= \mu_A^b(\{a_1, \ldots, a_j\}) \\ \varphi_B^a(i/n) &= \mu_B^a(\{b_1, \ldots, b_i\}) \end{aligned}$$

for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, $a \in A$, $b \in B$, subsets $\{a_1, \ldots, a_j\} \subseteq A$, $\{b_1, \ldots, b_i\} \subseteq B$, with $|A| = m$ and $|B| = n$.

Proposition 15. *In the framework of Definition 14, the quantified concept-*

forming operators $^{\uparrow_{A^Q}}\colon L_2^B \longrightarrow L_1^A$ *and* $^{\downarrow^{Q_B}}\colon L_1^A \longrightarrow L_2^B$*, satisfy:*

$$g^{\uparrow_{Q_B}}(a) \;=\; \bigvee_{i=1}^{n} \Big(R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)})\Big) \mathbin{\&} \varphi_B^a(i/n) \tag{10}$$

$$f^{\downarrow^{A^Q}}(b) \;=\; \bigvee_{j=1}^{m} \varphi_A^b(j/m) \mathbin{\&} \Big(R(a_{\lambda_b(j)}, b) \nwarrow_{\sigma(a_{\lambda_b(j)},b)} f(a_{\lambda_b(j)})\Big) \tag{11}$$

for all $g \in L_2^B$*,* $f \in L_1^A$ *and* $a \in A$*,* $b \in B$*, where* π_a *and* λ_b *are permutations such as:*

$$\begin{aligned} R(a, b_{\pi_a(i+1)}) \swarrow^{\sigma(a,b_{\pi_a(i+1)})} g(b_{\pi_a(i+1)}) &\leq R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)}) \\ R(a_{\lambda_b(j+1)}, b) \nwarrow_{\sigma(a_{\lambda_b(j+1)},b)} f(a_{\lambda_b(j+1)}) &\leq R(a_{\lambda_b(j)}, b) \nwarrow_{\sigma(a_{\lambda_b(j)},b)} f(a_{\lambda_b(j)}) \end{aligned}$$

Proof. The proof straightforwardly follows from Definition 14 and Theorem 8. □

As a consequence, Definition 14 provides a quantified version of the concept-forming operators, which reduces the existential character of the original ones. For example, given the fuzzy measures $\mu_B^a\colon \mathcal{B} \to [0,1]$, $\mu_A^b\colon \mathcal{A} \to [0,1]$ defined as

$$\mu_B^a(X) = \begin{cases} 1 & \text{if } X = B \text{ or } X = B \setminus \{b\}, \text{ with } b \in B \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_A^b(Y) = \begin{cases} 1 & \text{if } Y = A \text{ or } Y = A \setminus \{a\}, \text{ with } a \in A \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in A$, $b \in B$, $X \in \mathcal{P}(B)$, $Y \in \mathcal{P}(A)$. They clearly are invariant with respect to the cardinality and the associated fuzzy sets φ_B^a and φ_A^b are defined as

$$\varphi_B^a(i/n) \;=\; \begin{cases} 1 & \text{if } i = n, n-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_A^b(j/m) \;=\; \begin{cases} 1 & \text{if } j = m, m-1 \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, with $|B| = n$ and $|A| = m$. From Proposition 15, we can easily show that the consideration of these measures reduce the existentially character of the original definition removing the worst case, that is, the smallest value of the computation: $R(a, b_{\pi_a(n)}) \swarrow^{\sigma(a,b_{\pi_a(n)})} g(b_{\pi_a(n)})$ and $R(a_{\lambda_b(m)}, b) \nwarrow_{\sigma(a_{\lambda_b(m)},b)} f(a_{\lambda_b(m)})$, respectively. Therefore, for instance, instead of obtaining:

$$g^{\uparrow}(a) \;=\; \bigwedge_{b \in B} \Big(R(a, b) \swarrow^{\sigma(a,b)} g(b)\Big) = R(a, b_{\pi_a(n)}) \swarrow^{\sigma(a,b_{\pi_a(n)})} g(b_{\pi_a(n)})$$

R	b_1	b_2	b_3
a_1	0.75	1	0.75
a_2	0.25	0.5	0.75
a_3	0.25	0.5	0.5
a_4	0	0.75	1

Table 1: Relation R of Example 16

by the definition of the permutation π_a, we obtain that

$$\begin{aligned} g^{\uparrow_{Q_B}}(a) &= \bigvee_{i=1}^{n} \Big(R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)})\Big) \mathbin{\&} \varphi_B^a(i/n) \\ &= \bigvee_{i=n-1}^{n} \Big(R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)})\Big) \mathbin{\&} 1 \\ &= \bigvee_{i=n-1}^{n} \Big(R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)})\Big) \\ &= R(a, b_{\pi_a(n-1)}) \swarrow^{\sigma(a,b_{\pi_a(n-1)})} g(b_{\pi_a(n-1)}) \end{aligned}$$

where the first and second equalities follow from the boundary condition of $\&$ with respect to the bottom and top elements in $[0,1]$, and the last equality holds by the definition of π_a.

Example 16. Let $([0,1], \&_G, \&_L)$ be the multi-adjoint frame composed of the compete lattice $([0,1], \leq)$ and the Gödel and Łukasiewicz conjunctors, $\&_G$ and $\&_L$, respectively (for more details, see [16]). The considered context (A, B, R, σ) is composed of the set of attributes $A = \{a_1, a_2, a_3, a_4\}$, the set of objects $B = \{b_1, b_2, b_3\}$, the relation $R\colon A \times B \to [0,1]$ displayed in Table 16 and the mapping σ is defined, for all $a \in A$ and $b \in B$, as:

$$\sigma(a,b) = \begin{cases} a \mathbin{\&_L} b & \text{if } a = a_4 \\ a \mathbin{\&_G} b & \text{otherwise} \end{cases}$$

In order to illustrate the characterization of the conjunctive quantified concept-forming operators, we consider the product conjunctor $\&_P\colon [0,1] \times [0,1] \to [0,1]$ defined as $x \mathbin{\&_P} y = x * y$, for all $x, y \in [0,1]$, the fuzzy sets $g\colon B \to [0,1]$,

$\varphi^a_B\colon \{1,2,3\} \to \{0,1\}$ and $\varphi^b_A\colon \{1,2,3,4\} \to \{0,1\}$, which are defined[1] as:

$$
\begin{aligned}
g(b) &= \begin{cases} 1 & \text{if } b = b_3 \\ 0.5 & \text{otherwise} \end{cases} \\
\varphi^a_B(i/3) &= \begin{cases} 1 & \text{if } i \in \{2,3\} \\ 0 & \text{otherwise} \end{cases} \\
\varphi^b_A(j/4) &= \begin{cases} 1 & \text{if } j \in \{3,4\} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

for all $b \in B$, $a \in A$, and the following permutations $\pi_{a_1}, \pi_{a_2}, \pi_{a_3}, \pi_{a_4}\colon \{1,2,3\} \to \{1,2,3\}$ defined as:

$$
\begin{aligned}
\pi_{a_1}(i) &= \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \\ 3 & \text{if } i = 3 \end{cases} \\
\pi_{a_2}(i) = \pi_{a_3}(i) = \pi_{a_4}(i) &= \begin{cases} 2 & \text{if } i = 1 \\ 3 & \text{if } i = 2 \\ 1 & \text{if } i = 3 \end{cases}
\end{aligned}
$$

We fix the attribute a_1 and we check that the next inequality holds for $i = 1$ and $i = 2$:

$$R(a_1, b_{\pi_{a_1}(i+1)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(i+1)})} g(b_{\pi_{a_1}(i+1)}) \leq R(a_1, b_{\pi_{a_1}(i)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(i)})} g(b_{\pi_{a_1}(i)})$$

- Case $i = 1$.

$$
\begin{aligned}
R(a_1, b_{\pi_{a_1}(2)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(2)})} g(b_{\pi_{a_1}(2)}) &= R(a_1, b_1) \swarrow^{\sigma(a_1, b_1)} g(b_1) \\
&= 0.75 \swarrow^{\mathrm{G}} 0.5 \\
&\leq 1 \swarrow^{\mathrm{G}} 0.5 \\
&= R(a_1, b_2) \swarrow^{\sigma(a_1, b_2)} g(b_2) \\
&= R(a_1, b_{\pi_{a_1}(1)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(1)})} g(b_{\pi_{a_1}(1)})
\end{aligned}
$$

[1]Notice that, due to the definitions of φ^b_A and φ^a_B, any adjoint conjunctor could be considered satisfying the boundary condition with the top element.

- Case $i = 2$.

$$\begin{aligned} R(a_1, b_{\pi_{a_1}(3)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(3)})} g(b_{\pi_{a_1}(3)}) &= R(a_1, b_3) \swarrow^{\sigma(a_1, b_3)} g(b_3) \\ &= 0.75 \swarrow^{\mathrm{G}} 1 \\ &= 0.75 \\ &\leq 1 \\ &= 0.75 \swarrow^{\mathrm{G}} 0.5 \\ &= R(a_1, b_1) \swarrow^{\sigma(a_1, b_1)} g(b_1) \\ &= R(a_1, b_{\pi_{a_1}(2)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(2)})} g(b_{\pi_{a_1}(2)}) \end{aligned}$$

Analogously, it can be proven that the inequality

$$R(a, b_{\pi_a(i+1)}) \swarrow^{\sigma(a, b_{\pi_a(i+1)})} g(b_{\pi_a(i+1)}) \leq R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a, b_{\pi_a(i)})} g(b_{\pi_a(i)})$$

is satisfied for $i \in \{1, 2\}$, when $a \in \{a_2, a_3, a_4\}$.

Since the hypotheses required in Proposition 15 are satisfied, we can apply the characterization of the conjunctive quantified concept-forming operators, obtaining the following chain of equalities:

$$\begin{aligned} g^{\uparrow_{Q_B}}(a_1) &= \bigvee_{i=1}^{3} \Big(R(a_1, b_{\pi_{a_1}(i)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(i)})} g(b_{\pi_{a_1}(i)}) \Big) \,\&_{\mathrm{P}}\, \varphi_B^{a_1}(i/3) \\ &= \bigvee_{i=2}^{3} \Big(R(a_1, b_{\pi_{a_1}(i)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(i)})} g(b_{\pi_{a_1}(i)}) \Big) \,\&_{\mathrm{P}}\, 1 \\ &= \bigvee_{i=2}^{3} \Big(R(a_1, b_{\pi_{a_1}(i)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(i)})} g(b_{\pi_{a_1}(i)}) \Big) \\ &= R(a_1, b_{\pi_{a_1}(2)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(2)})} g(b_{\pi_{a_1}(2)}) \\ &= R(a_1, b_1) \swarrow^{\sigma(a_1, b_1)} g(b_1) \\ &= 0.75 \swarrow^{\mathrm{G}} 0.5 = 1 \end{aligned}$$

Notice that, the second and third equalities are deduced from the boundary condition of $\&_{\mathrm{P}}$ with respect to 0 and 1, and the fourth equality holds by the definition of π_{a_1}.

Following an analogous reasoning to the previous one, it is easy to check the following equalities:

$$\begin{aligned} g^{\uparrow_{Q_B}}(a_2) &= R(a_2, b_{\pi_{a_2}(2)}) \swarrow^{\sigma(a_2, b_{\pi_{a_2}(2)})} g(b_{\pi_{a_2}(2)}) \\ &= R(a_2, b_3) \swarrow^{\sigma(a_2, b_3)} g(b_3) \\ &= 0.75 \swarrow^{\mathrm{G}} 1 = 0.75 \end{aligned}$$

$$\begin{aligned} g^{\uparrow_{Q_B}}(a_3) &= R(a_3, b_{\pi_{a_3}(2)}) \swarrow^{\sigma(a_3, b_{\pi_{a_3}(2)})} g(b_{\pi_{a_3}(2)}) \\ &= R(a_3, b_3) \swarrow^{\sigma(a_3, b_3)} g(b_3) \\ &= 0.5 \swarrow^{\mathrm{G}} 1 = 0.5 \end{aligned}$$

$$
\begin{aligned}
g^{\uparrow_{Q_B}}(a_4) &= R(a_4, b_{\pi_{a_4}(2)}) \swarrow^{\sigma(a_4, b_{\pi_{a_4}(2)})} g(b_{\pi_{a_4}(2)}) \\
&= R(a_4, b_3) \swarrow^{\sigma(a_4, b_3)} g(b_3) \\
&= 1 \swarrow^{\mathrm{L}} 1 = 1
\end{aligned}
$$

Therefore, $g^{\uparrow_{Q_B}} = (a_1/1, a_2/0.75, a_3/0.5, a_4/1)$. As it was previously commented, these values correspond to the second smallest values in the computation of the original concept-forming operators (Definition 4).

$$
\begin{aligned}
g^{\uparrow}(a_1) &= \inf\{R(a_1, b) \swarrow^{\mathrm{G}} g(b) \mid b \in B\} = \inf\{1, 1, 0.75\} = 0.75 \\
g^{\uparrow}(a_2) &= \inf\{R(a_2, b) \swarrow^{\mathrm{G}} g(b) \mid b \in B\} = \inf\{0.25, 1, 0.75\} = 0.25 \\
g^{\uparrow}(a_3) &= \inf\{R(a_3, b) \swarrow^{\mathrm{G}} g(b) \mid b \in B\} = \inf\{0.25, 1, 0.5\} = 0.25 \\
g^{\uparrow}(a_4) &= \inf\{R(a_4, b) \swarrow^{\mathrm{L}} g(b) \mid b \in B\} = \inf\{0.5, 1, 1\} = 0.5
\end{aligned}
$$

Hence, the quantified concept-forming operators can correct some possible noise in the data. For example, the value $R(a_4, b_1) = 0$ could be given by the user because he/she did not know the relationship between a_4 and b_1 instead of that b_1 does not have attribute a_4. Hence, this error from the user will provide that $g^{\uparrow}(a_4) = 0.5$, however, it would be more convenient to consider $g^{\uparrow_{Q_B}}(a_4) = 1$, of course, taking into account that a value is not considered and we are obtaining conclusions without the consideration of all dataset. In this toy example with only three objects, the quantifier operators provide great changes, but in bigger datasets, this consideration is more convenient.

From now on, we consider the fuzzy set $f\colon A \to [0, 1]$ defined as

$$
f(a) = \begin{cases} 1 & \text{if } a \in \{a_1, a_4\} \\ 0.75 & \text{if } a = a_2 \\ 0.5 & \text{if } a = a_3 \end{cases}
$$

and the permutations $\lambda_{b_1}, \lambda_{b_2}, \lambda_{b_3}\colon \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ defined as:

$$
\lambda_{b_1}(j) = \begin{cases} 1 & \text{if } j = 1 \\ 3 & \text{if } j = 2 \\ 2 & \text{if } j = 3 \\ 4 & \text{if } j = 4 \end{cases}
\quad
\lambda_{b_2}(j) = \begin{cases} 1 & \text{if } j = 1 \\ 3 & \text{if } j = 2 \\ 4 & \text{if } j = 3 \\ 2 & \text{if } j = 4 \end{cases}
\quad
\lambda_{b_3}(j) = \begin{cases} 2 & \text{if } j = 1 \\ 3 & \text{if } j = 2 \\ 4 & \text{if } j = 3 \\ 1 & \text{if } j = 4 \end{cases}
$$

in order to compute $f^{\downarrow^{\mathrm{Q}}_{A}}(b)$, for all $b \in B$, by using Proposition 15. Fixed $b \in B$ and making simple computations, it can easily be checked that the following inequality holds, for all $b \in B$ and $j \in \{1, 2, 3\}$:

$$
R(a_{\lambda_b(j+1)}, b) \nwarrow_{\sigma(a_{\lambda_b(j+1)}, b)} f(a_{\lambda_b(j+1)}) \quad \leq \quad R(a_{\lambda_b(j)}, b) \nwarrow_{\sigma(a_{\lambda_b(j)}, b)} f(a_{\lambda_b(j)})
$$

Fixed the object b_1, under the conditions of Proposition 15, we can apply the characterization of the conjunctive quantified concept-forming operators.

Taking into account the boundary condition of $\&_{\mathrm{P}}$ with respect to 0 and 1 and the definition of the permutation λ_{b_1}, we deduce the following chain of equalities:

$$\begin{aligned}
f^{\downarrow^{A^{\mathrm{Q}}}}(b_1) &= \bigvee_{j=1}^{4} \varphi_A^{b_1}(j/4) \,\&_{\mathrm{P}} \left(R(a_{\lambda_{b_1}(j)}, b_1) \nwarrow_{\sigma(a_{\lambda_{b_1}(j)}, b_1)} f(a_{\lambda_{b_1}(j)}) \right) \\
&= \bigvee_{j=3}^{4} 1 \;\;\&_{\mathrm{P}} \left(R(a_{\lambda_{b_1}(j)}, b_1) \nwarrow_{\sigma(a_{\lambda_{b_1}(j)}, b_1)} f(a_{\lambda_{b_1}(j)}) \right) \\
&= \bigvee_{j=3}^{4} \left(R(a_{\lambda_{b_1}(j)}, b_1) \nwarrow_{\sigma(a_{\lambda_{b_1}(j)}, b_1)} f(a_{\lambda_{b_1}(j)}) \right) \\
&= R(a_{\lambda_{b_1}(3)}, b_1) \nwarrow_{\sigma(a_{\lambda_{b_1}(3)}, b_1)} f(a_{\lambda_{b_1}(3)}) \\
&= R(a_2, b_1) \nwarrow_{\sigma(a_2, b_1)} f(a_2) \\
&= 0.25 \nwarrow_{\mathrm{G}} 0.75 = 0.25
\end{aligned}$$

Analogously, we compute $f^{\downarrow^{A^{\mathrm{Q}}}}(b_2)$ and $f^{\downarrow^{A^{\mathrm{Q}}}}(b_3)$ as follows:

$$\begin{aligned}
f^{\downarrow^{A^{\mathrm{Q}}}}(b_2) &= R(a_{\lambda_{b_2}(3)}, b_2) \nwarrow_{\sigma(a_{\lambda_{b_2}(3)}, b_2)} f(a_{\lambda_{b_2}(3)}) \\
&= R(a_4, b_2) \nwarrow_{\sigma(a_4, b_2)} f(a_4) \\
&= 0.75 \nwarrow_{\mathrm{L}} 1 = 0.75
\end{aligned}$$

$$\begin{aligned}
f^{\downarrow^{A^{\mathrm{Q}}}}(b_3) &= R(a_{\lambda_{b_3}(3)}, b_3) \nwarrow_{\sigma(a_{\lambda_{b_3}(3)}, b_3)} f(a_{\lambda_{b_3}(3)}) \\
&= R(a_4, b_3) \nwarrow_{\sigma(a_4, b_3)} f(a_4) \\
&= 1 \nwarrow_{\mathrm{L}} 1 = 1
\end{aligned}$$

Thus, $f^{\downarrow^{A^{\mathrm{Q}}}} = (b_1/0.25, b_2/0.75, b_3/1)$. This example illustrates that the fuzzy sets φ_B and φ_A, which are used in the definition of the conjunctive quantified concept-forming operators given in Proposition 15, reduce the existential character of the original definition. Specifically, the smallest values involved in the computations of $g^{\uparrow_{\mathrm{Q}_B}}(a)$ and $f^{\downarrow^{A^{\mathrm{Q}}}}(b)$ are removed, that is $R(a, b_{\pi_a(3)}) \swarrow^{\sigma(a, b_{\pi_a(3)})} g(b_{\pi_a(3)})$ and $R(a_{\lambda_b(4)}, b) \nwarrow_{\sigma(a_{\lambda_b(4)}, b)} f(a_{\lambda_b(4)})$, respectively. It is also important to mention that the use of the characterization of the conjunctive quantified concept-forming operators (Proposition 15), instead of the operators given in Definition 14, remarkably reduce the calculations. $\square$

Now, we will study the dual definition with respect to the implicative generalized quantifiers. Depending of the selection of implications $\swarrow$ and $\nwarrow$, we also have four possibilities to define the implicative quantified concept-forming operators. Next, we consider one of these possibilities and the rest can analogously be defined.

Definition 17. Given a multi-adjoint frame and a context for that frame, $\{\mu_A^b \mid b \in B\}$, $\{\mu_B^a \mid a \in A\}$ two families of fuzzy measures on A and B, respectively, which are invariant with respect to the cardinality and Q_A^b, Q_B^a the quantifiers determined by the fuzzy measures μ_A^b and μ_B^a, respectively, the *implicative quantified concept-forming operators* are denoted as ${}^{\uparrow_{\mathrm{IQ}}}\colon L_2^B \longrightarrow L_1^A$ and ${}^{\downarrow^{\mathrm{IQ}}}\colon L_1^A \longrightarrow L_2^B$, where L_2^B and L_1^A denote the set of fuzzy subsets $g\colon B \to L_2$ and $f\colon A \to L_1$, respectively, and are defined, for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, as:

$$g^{\uparrow_{\mathrm{IQ}}}(a) \;=\; \bigvee_{X \in \mathcal{P}(B)} \left(\bigwedge_{b \in X} R(a,b) \swarrow^{\sigma(a,b)} g(b) \right) \swarrow \; \mu_B^a(X) \tag{12}$$

$$f^{\downarrow^{\mathrm{IQ}}}(b) \;=\; \bigvee_{Y \in \mathcal{P}(A)} \left(\bigwedge_{a \in Y} R(a,b) \nwarrow_{\sigma(a,b)} f(a) \right) \nwarrow \; \mu_A^b(Y) \tag{13}$$

As in the conjunctive case, Corollary 9 provides the following characterization of the implicative quantified concept-forming operators, where the invariant with respect to the cardinality fuzzy measures μ_A^b and μ_B^a are associated with the fuzzy sets $\varphi_A^b\colon \{1,\dots,m\} \to [0,1]$ and $\varphi_B^a\colon \{1,\dots,n\} \to [0,1]$, where $|A| = m$ and $|B| = n$.

Proposition 18. *In the framework of Definition 17, the quantified concept-forming operators* ${}^{\uparrow_{IQ}}\colon L_2^B \longrightarrow L_1^A$ *and* ${}^{\downarrow^{IQ}}\colon L_1^A \longrightarrow L_2^B$*, satisfy:*

$$g^{\uparrow_{IQ}}(a) \;=\; \bigwedge_{i=1}^{n} \Big(R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)}) \Big) \swarrow \; \varphi_B^a(i/n) \tag{14}$$

$$f^{\downarrow^{IQ}}(b) \;=\; \bigwedge_{j=1}^{m} \Big(R(a_{\lambda_b(j)}, b) \nwarrow_{\sigma(a_{\lambda_b(j)},b)} f(a_{\lambda_b(j)}) \Big) \nwarrow \; \varphi_A^b(j/m) \tag{15}$$

for all $g \in L_2^B$*,* $f \in L_1^A$ *and* $a \in A$*,* $b \in B$*, where* π_a *and* λ_b *are permutations such as:*

$$R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)}) \;\le\; R(a, b_{\pi_a(i+1)}) \swarrow^{\sigma(a,b_{\pi_a(i+1)})} g(b_{\pi_a(i+1)})$$
$$R(a_{\lambda_b(j)}, b) \nwarrow_{\sigma(a_{\lambda_b(j)},b)} f(a_{\lambda_b(j)}) \;\le\; R(a_{\lambda_b(j+1)}, b) \nwarrow_{\sigma(a_{\lambda_b(j+1)},b)} f(a_{\lambda_b(j+1)})$$

PROOF. The equalities clearly hold by Definition 17 and Corollary 9.

As a consequence, Definition 17 provides a new quantified version of the concept-forming operators, which reduces the existential character of the original ones using, in this case, the implicative type. Although the conjunctive and implicative quantified concept-forming operators are different and provide different results, in general, they can coincide in some particular cases. For example, from Proposition 18, we can easily show that the consideration of the measures $\mu_B^a\colon \mathcal{P}(B) \to [0,1]$, $\mu_A^b\colon \mathcal{P}(A) \to [0,1]$ defined as

$$\mu_B^a(X) = \begin{cases} 0 & \text{if } X = \varnothing \text{ or } X = \{b\}, \text{ with } b \in B \\ 1 & \text{otherwise} \end{cases}$$

$$\mu_A^b(Y) = \begin{cases} 0 & \text{if } Y = \varnothing \text{ or } Y = \{a\}, \text{ with } a \in A \\ 1 & \text{otherwise} \end{cases}$$

for all $a \in A$, $b \in B$, $X \in \mathcal{P}(B)$, $Y \in \mathcal{P}(A)$, also corresponds to remove the worst case, that is, the smallest value of the computation: $R(a, b_{\pi(1)}) \swarrow^{\sigma(a,b_{\pi(1)})} g(b_{\pi(1)})$ and $R(a_{\lambda(1)}, b) \nwarrow_{\sigma(a_{\lambda(1)},b)} f(a_{\lambda(1)})$, respectively. Notice that fuzzy sets φ_A^b and φ_B^a associated with these measures are defined as

$$\begin{aligned}
\varphi_A^b(j/m) &= \begin{cases} 0 & \text{if } j = 1 \\ 1 & \text{otherwise} \end{cases} \\
\varphi_B^a(i/n) &= \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{otherwise} \end{cases}
\end{aligned}$$

for all $a \in A$, $b \in B$, $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$, with $|A| = m$ and $|B| = n$.

We will consider this case to illustrate the characterization of the implicative quantified concept-forming operators given in Proposition 18 in the following example.

Example 19. Given the multi-adjoint frame and context of Example 16, we will consider the same fuzzy sets g, and the fuzzy set φ_B^a and φ_A^b defined as:

$$\begin{aligned}
\varphi_B^a(i/3) &= \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \\
\varphi_A^b(j/4) &= \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

In addition, the implications $\swarrow^{\mathrm{P}}$ and $\nwarrow_{\mathrm{P}}$ associated with the product conjunctor $\&_P$ will be the implications $\swarrow$ and $\nwarrow$, respectively, used in Proposition 18. Notice that, $\swarrow^{\mathrm{P}} = \nwarrow_{\mathrm{P}}$ since the product conjunctor is commutative. For more details, see [16].

Since the same context and mappings have been considered, now the permutations are the opposite of the ones considered in Example 16. Hence, the permutations $\pi_{a_1}, \pi_{a_2}, \pi_{a_3}, \pi_{a_4} \colon \{1, 2, 3\} \to \{1, 2, 3\}$ are defined as follows:

$$\begin{aligned}
\pi_{a_1}(i) &= \begin{cases} 3 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \\ 2 & \text{if } i = 3 \end{cases} \\
\pi_{a_2}(i) = \pi_{a_3}(i) = \pi_{a_4}(i) &= \begin{cases} 1 & \text{if } i = 1 \\ 3 & \text{if } i = 2 \\ 2 & \text{if } i = 3 \end{cases}
\end{aligned}$$

Therefore, the conditions required in Proposition 18 are fullfilled and we can apply the characterization of the implicative quantified concept-forming operators. Specifically, we have that:

$$
\begin{aligned}
g^{\uparrow_{\mathrm{IQ}}}(a_1) &= \bigwedge_{i=1}^{3} \Big(R(a_1, b_{\pi_{a_1}(i)}) \swarrow^{\sigma(a_1, b_{\pi_{a_1}(i)})} g(b_{\pi_{a_1}(i)})\Big) \swarrow^{\mathrm{P}} \ \varphi_B^{a_1}(i/3) \\
&= \inf\{\big(0.75 \swarrow^{\mathrm{G}} 1\big) \swarrow^{\mathrm{P}} 0, \big(0.75 \swarrow^{\mathrm{G}} 0.5\big) \swarrow^{\mathrm{P}} 1, \big(1 \swarrow^{\mathrm{G}} 0.5\big) \swarrow^{\mathrm{P}} 1\} \\
&= \inf\{0.75 \swarrow^{\mathrm{P}} 0, 1 \swarrow^{\mathrm{P}} 1, 1 \swarrow^{\mathrm{P}} 1\} = \inf\{1,1,1\} = 1 \\
g^{\uparrow_{\mathrm{IQ}}}(a_2) &= \bigwedge_{i=1}^{3} \Big(R(a_2, b_{\pi_{a_2}(i)}) \swarrow^{\sigma(a_2, b_{\pi_{a_2}(i)})} g(b_{\pi_{a_2}(i)})\Big) \swarrow^{\mathrm{P}} \ \varphi_B^{a_2}(i/3) \\
&= \inf\{\big(0.25 \swarrow^{\mathrm{G}} 0.5\big) \swarrow^{\mathrm{P}} 0, \big(0.75 \swarrow^{\mathrm{G}} 1\big) \swarrow^{\mathrm{P}} 1, \big(0.5 \swarrow^{\mathrm{G}} 0.5\big) \swarrow^{\mathrm{P}} 1\} \\
&= \inf\{0.25 \swarrow^{\mathrm{P}} 0, 0.75 \swarrow^{\mathrm{P}} 1, 1 \swarrow^{\mathrm{P}} 1\} = \inf\{1, 0.75, 1\} = 0.75 \\
g^{\uparrow_{\mathrm{IQ}}}(a_3) &= \bigwedge_{i=1}^{3} \Big(R(a_3, b_{\pi_{a_3}(i)}) \swarrow^{\sigma(a_3, b_{\pi_{a_3}(i)})} g(b_{\pi_{a_3}(i)})\Big) \swarrow^{\mathrm{P}} \ \varphi_B^{a_3}(i/3) \\
&= \inf\{\big(0.25 \swarrow^{\mathrm{G}} 0.5\big) \swarrow^{\mathrm{P}} 0, \big(0.5 \swarrow^{\mathrm{G}} 1\big) \swarrow^{\mathrm{P}} 1, \big(0.5 \swarrow^{\mathrm{G}} 0.5\big) \swarrow^{\mathrm{P}} 1\} \\
&= \inf\{0.25 \swarrow^{\mathrm{P}} 0, 0.5 \swarrow^{\mathrm{P}} 1, 1 \swarrow^{\mathrm{P}} 1\} = \inf\{1, 0.5, 1\} = 0.5 \\
g^{\uparrow_{\mathrm{IQ}}}(a_4) &= \bigwedge_{i=1}^{3} \Big(R(a_4, b_{\pi_{a_4}(i)}) \swarrow^{\sigma(a_4, b_{\pi_{a_4}(i)})} g(b_{\pi_{a_4}(i)})\Big) \swarrow^{\mathrm{P}} \ \varphi_B^{a_4}(i/3) \\
&= \inf\{\big(0 \swarrow^{\mathrm{L}} 0.5\big) \swarrow^{\mathrm{P}} 0, \big(1 \swarrow^{\mathrm{L}} 1\big) \swarrow^{\mathrm{P}} 1, \big(0.75 \swarrow^{\mathrm{L}} 0.5\big) \swarrow^{\mathrm{P}} 1\} \\
&= \inf\{0.5 \swarrow^{\mathrm{P}} 0, 1 \swarrow^{\mathrm{P}} 1, 1 \swarrow^{\mathrm{P}} 1\} = \inf\{1,1,1\} = 1
\end{aligned}
$$

Notice that $g^{\uparrow_{\mathrm{IQ}}} = (a_1/1, a_2/0.75, a_3/0.5, a_4/1)$ coincides with $g^{\uparrow_{\mathrm{Q}_B}}$ computed in Example 16, as we previously observed. Now, we show that $f^{\downarrow^{\mathrm{IQ}}}(b)$ also is equal to $f^{\downarrow^{A^{\mathrm{Q}}}}(b)$ (computed in Example 16). For that purpose, we consider the fuzzy set f given in Example 16 and the opposite permutations $\lambda_{b_1}, \lambda_{b_2}, \lambda_{b_3} \colon \{1,2,3,4\} \to \{1,2,3,4\}$, that is:

$$
\lambda_{b_1}(j) = \begin{cases} 4 & \text{if } j = 1 \\ 2 & \text{if } j = 2 \\ 3 & \text{if } j = 3 \\ 1 & \text{if } j = 4 \end{cases} \quad \lambda_{b_2}(j) = \begin{cases} 2 & \text{if } j = 1 \\ 4 & \text{if } j = 2 \\ 3 & \text{if } j = 3 \\ 1 & \text{if } j = 4 \end{cases} \quad \lambda_{b_3}(j) = \begin{cases} 1 & \text{if } j = 1 \\ 4 & \text{if } j = 2 \\ 3 & \text{if } j = 3 \\ 2 & \text{if } j = 4 \end{cases}
$$

Fixed $b \in B$, it can be checked in an easy way that the next inequality

$$
R(a_{\lambda_b(j)}, b) \nwarrow_{\sigma(a_{\lambda_b(j)}, b)} f(a_{\lambda_b(j)}) \quad \leq \quad R(a_{\lambda_b(j+1)}, b) \nwarrow_{\sigma(a_{\lambda_b(j+1)}, b)} f(a_{\lambda_b(j+1)})
$$

holds, for all $a \in A$ and $j \in \{1,2,3\}$. Hence, applying Proposition 18, we obtain:

$$\begin{aligned}
f^{\downarrow^{\mathrm{IQ}}}(b_1) &= \bigwedge_{j=1}^{4} \Big(R(a_{\lambda_{b_1}(j)}, b_1) \nwarrow_{\sigma(a_{\lambda_{b_1}(j)}, b_1)} f(a_{\lambda_{b_1}(j)})\Big) \nwarrow_{\mathrm{P}} \ \varphi_A^{b_1}(j/4) \\
&= \inf\{(0 \nwarrow_{\mathrm{L}} 1) \nwarrow_{\mathrm{P}} 0, (0.25 \nwarrow_{\mathrm{G}} 0.75) \nwarrow_{\mathrm{P}} 1, (0.25 \nwarrow_{\mathrm{G}} 0.5) \nwarrow_{\mathrm{P}} 1, (0.75 \nwarrow_{\mathrm{G}} 1) \nwarrow_{\mathrm{P}} 1\} \\
&= \inf\{0 \nwarrow_{\mathrm{P}} 0, 0.25 \nwarrow_{\mathrm{P}} 1, 0.25 \nwarrow_{\mathrm{P}} 1, 0.75 \nwarrow_{\mathrm{P}} 1\} \\
&= \inf\{1, 0.25, 0.25, 0.75\} = 0.25 \\
f^{\downarrow^{\mathrm{IQ}}}(b_2) &= \bigwedge_{j=1}^{4} \Big(R(a_{\lambda_{b_2}(j)}, b_2) \nwarrow_{\sigma(a_{\lambda_{b_2}(j)}, b_2)} f(a_{\lambda_{b_2}(j)})\Big) \nwarrow_{\mathrm{P}} \ \varphi_A^{b_2}(j/4) \\
&= \inf\{(0.5 \nwarrow_{\mathrm{G}} 0.75) \nwarrow_{\mathrm{P}} 0, (0.75 \nwarrow_{\mathrm{L}} 1) \nwarrow_{\mathrm{P}} 1, (0.5 \nwarrow_{\mathrm{G}} 0.5) \nwarrow_{\mathrm{P}} 1, (1 \nwarrow_{\mathrm{G}} 1) \nwarrow_{\mathrm{P}} 1\} \\
&= \inf\{0.5 \nwarrow_{\mathrm{P}} 0, 0.75 \nwarrow_{\mathrm{P}} 1, 1 \nwarrow_{\mathrm{P}} 1, 1 \nwarrow_{\mathrm{P}} 1\} \\
&= \inf\{1, 0.75, 1, 1\} = 0.75 \\
f^{\downarrow^{\mathrm{IQ}}}(b_3) &= \bigwedge_{j=1}^{4} \Big(R(a_{\lambda_{b_3}(j)}, b_3) \nwarrow_{\sigma(a_{\lambda_{b_3}(j)}, b_3)} f(a_{\lambda_{b_3}(j)})\Big) \nwarrow_{\mathrm{P}} \ \varphi_A^{b_3}(j/4) \\
&= \inf\{(0.75 \nwarrow_{\mathrm{G}} 1) \nwarrow_{\mathrm{P}} 0, (1 \nwarrow_{\mathrm{L}} 1) \nwarrow_{\mathrm{P}} 1, (0.5 \nwarrow_{\mathrm{G}} 0.5) \nwarrow_{\mathrm{P}} 1, (0.75 \nwarrow_{\mathrm{G}} 0.75) \nwarrow_{\mathrm{P}} 1\} \\
&= \inf\{0.75 \nwarrow_{\mathrm{P}} 0, 1 \nwarrow_{\mathrm{P}} 1, 1 \nwarrow_{\mathrm{P}} 1, 1 \nwarrow_{\mathrm{P}} 1\} \\
&= \inf\{1, 1, 1, 1\} = 1
\end{aligned}$$

Thus, $f^{\downarrow^{\mathrm{IQ}}} = (b_1/0.25, b_2/0.75, b_3/1)$, which is equal to $f^{\downarrow^{A\mathrm{Q}}}$, computed in Example 16. As we mentioned previously, in general, both quantified concept-forming operators are different and provide different point of views. □

The following section relates the implicative quantified concept-forming operators with a well-know framework, which has been used in the reduction procedure of concept lattice.

4.1. Comparison with threshold concept lattices

In the literature there exists a philosophy, whose concept forming operators have a similar syntax to the ones given here with implicative generalized quantifiers. Variable threshold concept lattices [7, 19, 45] consider a threshold for normalizing the concept-forming operators used in different fuzzy concept lattice frameworks [19]. This section will recall the definitions given in [19] on the unit interval and will show a comparison with the definitions introduced above.

Definition 20 ([19]). Given two thresholds $\delta_1, \delta_2 \in [0,1]$, the *threshold concept-forming operators* ${}^{\uparrow_{\delta_2}} \colon L^B \longrightarrow L^A$ and ${}^{\downarrow^{\delta_1}} \colon L^A \longrightarrow L^B$, are defined as following

$$g^{\uparrow_{\delta_2}}(a) = \bigwedge_{b \in B} \Big(R(a,b) \swarrow^{\sigma(a,b)} g(b)\Big) \swarrow \delta_2 \tag{16}$$

$$f^{\downarrow^{\delta_1}}(b) = \bigwedge_{a \in A} \big(R(a,b) \nwarrow_{\sigma(a,b)} f(a)\big) \nwarrow \delta_1 \tag{17}$$

for all $g \in L^B$, $f \in L^A$ and $a \in A$, $b \in B$.

Other combinations of the implications $\swarrow$ and $\nwarrow$ were also studied in [19] and, as a conclusion, the authors proved that the case given above has better properties and so, it is the most suitable to be considered. The following result shows the relationship with operators given in Definition 17.

Proposition 21. *In the framework of Definition 17, given $\delta_1, \delta_2 \in [0,1]$, the fuzzy measures*

$$\mu_B^{a,\delta_2}(X) = \begin{cases} 1 & \text{if } X = B \\ 0 & \text{if } X = \varnothing \\ \delta_2 & \text{otherwise} \end{cases} \qquad \mu_A^{b,\delta_1}(Y) = \begin{cases} 1 & \text{if } Y = A \\ 0 & \text{if } Y = \varnothing \\ \delta_1 & \text{otherwise} \end{cases}$$

and the fuzzy sets $g \in L^B$, $f \in L^A$. If the following inequalities hold for all $a \in A$, $b \in B$,

$$\Big(R(a, b_{\pi_a(1)}) \swarrow^{\sigma(a,b_{\pi_a(1)})} g(b_{\pi_a(1)})\Big) \swarrow \delta_2 \quad \leq \quad R(a, b_{\pi_a(n)}) \swarrow^{\sigma(a,b_{\pi_a(n)})} g(b_{\pi_a(n)})$$

$$\Big(R(a_{\lambda_b(1)}, b) \nwarrow_{\sigma(a_{\lambda_b(1)},b)} f(a_{\lambda_b(1)})\Big) \nwarrow \delta_1 \quad \leq \quad R(a_{\lambda_b(m)}, b) \nwarrow_{\sigma_b(a_{\lambda_b(m)},b)} f(a_{\lambda_b(m)})$$

were π_a and λ_b are permutations satisfying:

$$\begin{aligned} R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)}) \quad &\leq \quad R(a, b_{\pi_a(i+1)}) \swarrow^{\sigma(a,b_{\pi_a(i+1)})} g(b_{\pi_a(i+1)}) \\ R(a_{\lambda_b(j)}, b) \nwarrow_{\sigma(a_{\lambda_b(j)},b)} f(a_{\lambda_b(j)}) \quad &\leq \quad R(a_{\lambda_b(j+1)}, b) \nwarrow_{\sigma(a_{\lambda_b(j+1)},b)} f(a_{\lambda_b(j+1)}) \end{aligned}$$

then we obtain that

$$g^{\uparrow_{IQ}} = g^{\uparrow_{\delta_2}} \qquad f^{\downarrow^{IQ}} = f^{\downarrow^{\delta_1}}$$

Proof. Given $g \in L_2^B$, we will prove the first equality. The other one holds similarly. Since $\swarrow$ is increasing in the consequent, by the definition of π_a, we have that

$$\Big(R(a, b_{\pi_a(1)}) \swarrow^{\sigma(a,b_{\pi_a(1)})} g(b_{\pi(1)})\Big) \swarrow \delta_2 \leq \Big(R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)})\Big) \swarrow \delta_2 \tag{18}$$

for all $i \in \{1, \ldots, n\}$. Moreover, the mapping φ_B^a associated with the fuzzy measure μ_B^{a,δ_2} satisfies that $\varphi_B^a(1) = 1$ and $\varphi_B^a(i/n) = \delta_2$, for all $i \in \{1, \ldots, n-1\}$. Therefore, for every $a \in A$ we have

$$\begin{aligned} g^{\uparrow_{\mathrm{IQ}}}(a) \quad &= \quad \bigwedge_{i=1}^{n} \Big(R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)})\Big) \swarrow \ \varphi_B^a(i/n) \\ &= \quad \left(\bigwedge_{i=1}^{n-1} \Big(R(a, b_{\pi_a(i)}) \swarrow^{\sigma(a,b_{\pi_a(i)})} g(b_{\pi_a(i)})\Big) \swarrow \delta_2\right) \bigwedge \Big(R(a, b_{\pi_a(n)}) \swarrow^{\sigma(a,b_{\pi_a(n)})} g(b_{\pi_a(n)})\Big) \\ &\overset{(1)}{=} \quad \Big(R(a, b_{\pi_a(1)}) \swarrow^{\sigma(a,b_{\pi_a(1)})} g(b_{\pi_a(1)})\Big) \swarrow \delta_2 \bigwedge \Big(R(a, b_{\pi_a(n)}) \swarrow^{\sigma(a,b_{\pi_a(n)})} g(b_{\pi_a(n)})\Big) \\ &\overset{(2)}{=} \quad \Big(R(a, b_{\pi_a(1)}) \swarrow^{\sigma(a,b_{\pi_a(1)})} g(b_{\pi_a(1)})\Big) \swarrow \delta_2 \\ &\overset{(3)}{=} \quad \bigwedge_{b \in B} \Big(R(a, b) \swarrow^{\sigma(a,b)} g(b)\Big) \swarrow \delta_2 \\ &= \quad g^{\uparrow_{\delta_2}}(a) \end{aligned}$$

where (1) holds by Equation (18), (2) by hypothesis and (3) by the definition of π_a and Equation (18). $\square$

This result and the definitions of ${}^{\uparrow_{\delta_2}}$ and ${}^{\downarrow^{\delta_1}}$ displays and justifies the possibility of considering in the definition of generalized quantifiers non-normalized measures, that is, we can take into account increasing mappings $\tau\colon \mathcal{P}(\mathcal{U}) \to [0,1]$, with $\tau(\varnothing) = 0$ and invariant with respect to the cardinality, without the requirement that $\tau(\mathcal{U}) = 1$. Clearly, these mappings are also bounded by the maximum fuzzy measure, i.e. $\tau \leq \mu_\exists$. In that case, we can consider the mappings

$$\tau_B^{a,\delta_2}(X) = \begin{cases} 0 & \text{if } X = \varnothing \\ \delta_2 & \text{otherwise} \end{cases} \qquad \tau_A^{b,\delta_1}(D) = \begin{cases} 0 & \text{if } Y = \varnothing \\ \delta_1 & \text{otherwise} \end{cases}$$

and Proposition 21 is rewriting as follows.

Proposition 22. *In the framework of Definition 17, considering the mappings* τ_B^{a,δ_2} *and* τ_A^{b,δ_1}*, we have that*

$$g^{\uparrow_{IQ}} = g^{\uparrow_{\delta_2}} \qquad f^{\downarrow^{IQ}} = f^{\downarrow^{\delta_1}}$$

for all $g \in L_2^B$*,* $f \in L_1^A$*.*

Proof. The proof follows similarly to the one given to Proposition 21 taking into account that in this case the mappings τ_B^{a,δ_2} and τ_A^{b,δ_1} are equal to δ_2 and δ_1 on B and A, respectively, instead of 1. $\square$

Therefore the threshold concept-forming operators can be seen as particular cases of implicative quantifier concept-forming operators. This fact also highlights the relevance of the new operators considered in this paper.

4.2. Properties of quantifier formal-concept operators

The first property we highlight is the monotonicity of both, the conjunctive and implicative quantified concept-forming operators. Since & is monotonic, then the conjunctive quantified concept-forming operators clearly are monotonic. Moreover, due to the implicative quantifiers evaluate the fuzzy set on the consequent and the implication is increasing in this argument, then the implicative quantified concept-forming operators also are monotonic.

The following result shows that the pair ${}^{\uparrow_{Q_A}}$ and ${}^{\downarrow^{Q_B}}$ form an antitone Galois connection, when the minimum fuzzy measure $\mu_\forall$ is considered.

Proposition 23. *Given the quantifiers* ${}_A^\forall Q$ *and* $Q_B^\forall$ *determined by the minimum fuzzy measure* $\mu_\forall$ *on the universes* A *and* B*, respectively, then the pair* $({}^{\uparrow_{Q_B^\forall}}, {}^{\uparrow_{{}_A^\forall Q}})$ *is an antitone Galois connection.*

Proof. The proof holds because the universal quantifier provides the original definitions of the concept-forming operators, that is, ${}^{\uparrow_{Q^{\forall}_{B}}} = {}^{\uparrow}$ and ${}^{\uparrow^{\forall}_{A}\mathrm{Q}} = {}^{\downarrow}$.

Due to the permutations π_a and λ_b in Definition 14, we obtain that $R(a, b_{\pi_a(n)}) \swarrow^{\sigma(a,b_{\pi_a(n)})} g(b_{\pi_a(n)})$ and $R(a_{\lambda_b(m)}, b) \nwarrow_{\sigma(a_{\lambda_b(m)},b)} f(a_{\lambda_b(m)})$ are the least elements of the sets

$$\{R(a,b) \swarrow^{\sigma(a,b)} g(b) \mid b \in B\}$$
$$\{R(a,b) \nwarrow_{\sigma(a,b)} f(a) \mid a \in A\}$$

respectively. Therefore, ${}^{\uparrow_{Q^{\forall}_{B}}} = {}^{\uparrow}$ and ${}^{\uparrow^{\forall}_{A}\mathrm{Q}} = {}^{\downarrow}$. $\square$

Notice also that the pair $({}^{\uparrow_{\delta_2}}, {}^{\downarrow^{\delta_1}})$ is not a Galois connection in general. In [19], different properties of ${}^{\uparrow_{\delta_2}}$ and ${}^{\downarrow^{\delta_1}}$ were studied, where diverse ones were focused on providing sufficient conditions to ensure that ${}^{\uparrow_{\delta_2}}$ and ${}^{\downarrow^{\delta_1}}$ form a Galois connection. For example, when $\delta_1 = \delta_2$ and the conjunctor operators are associative, then we can assert that $({}^{\uparrow_{\delta_1}}, {}^{\downarrow^{\delta_1}})$ is a Galois connection. For instance, these hypotheses hold in the variable threshold frameworks considered in [7, 45], in which residuated lattices are considered.

Hence, clearly, in the framework of Proposition 22, the pair $({}^{\uparrow_{\mathrm{IQ}}}, {}^{\downarrow^{\mathrm{IQ}}})$ is a Galois connection. The following example shows that this fact does not hold when the fuzzy measures considered in Proposition 21 are considered.

Example 24. Given the frame $([0,1], \&_{\mathrm{G}})$, where $\&_G$ is the Gödel t-norm, the context (A, B, R, σ), where $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, R is defined in Table 2 and σ is constantly $\&_{\mathrm{G}}$, and the mappings $g\colon B \to [0,1]$, $f\colon A \to [0,1]$, defined as $g(b_1) = 0.8$, $g(b_2) = 0.8$, $f(a_1) = 1$, $f(a_2) = 0.8$.

Table 2: Relation R of the context of Example 24.

R	b_1	b_2
a_1	0.4	1
a_2	0.7	0.8

Moreover, we consider the following fuzzy measures

$$\mu_B^{a,0.4}(X) = \begin{cases} 1 & \text{if } X = B \\ 0 & \text{if } X = \varnothing \\ 0.4 & \text{otherwise} \end{cases} \qquad \mu_A^{b,0.4}(Y) = \begin{cases} 1 & \text{if } Y = A \\ 0 & \text{if } Y = \varnothing \\ 0.4 & \text{otherwise} \end{cases}$$

for all $a \in A$ and $b \in B$. We will prove that the adjoint property does not hold in this case, that is, we will see that $f \leq g^{\uparrow_{\mathrm{IQ}}}$, but $g \not\leq f^{\downarrow^{\mathrm{IQ}}}$.

Since $R(a_1, b_1) \swarrow^{\mathrm{G}} g(b_1) = 0.4 \swarrow^{\mathrm{G}} 0.8 = 0.4$, and $R(a_1, b_2) \swarrow^{\mathrm{G}} g(b_2) = 1 \swarrow^{\mathrm{G}} 0.9 = 1$, we have that

$$g^{\uparrow_{\mathrm{IQ}}}(a_1) = \inf\{0.4 \swarrow^{\mathrm{G}} 0.4, 1 \swarrow^{\mathrm{G}} 1\} = 1$$

Analogously, we obtain that

$$g^{\uparrow_{\mathrm{IQ}}}(a_2) = \inf\{0.7 \swarrow^{\mathrm{G}} 0.4, 0.8 \swarrow^{\mathrm{G}} 1\} = 0.8$$

Hence, $f \le g^{\uparrow_{\mathrm{IQ}}}$. On the other hand, we have

$$\begin{aligned} f^{\downarrow^{\mathrm{IQ}}}(b_1) &= \{0.4 \nwarrow_{\mathrm{G}} 0.4, 0.7 \nwarrow_{\mathrm{G}} 1\} = 0.7 \\ f^{\downarrow^{\mathrm{IQ}}}(b_2) &= \{1 \nwarrow_{\mathrm{G}} 0.4, 1 \nwarrow_{\mathrm{G}} 1\} = 1 \end{aligned}$$

Thus, $g \not\le f^{\downarrow^{\mathrm{IQ}}}$.

Example 16 provides a pair of conjunctive quantified concept-forming operators, which does not form a Galois connection. In this example, we have that $g = (b_1/0.5, b_2/0.5, b_3/1) \not\le (b_1/0.25, b_2/0.5, b_3/1) = g^{\uparrow_{A^{\mathrm{Q}}}\downarrow^{A^{\mathrm{Q}}}}$.

Hence, as a consequence, in general the quantifiers concept-forming operators do not form a Galois connection. The following example shows an example of Galois connections.

Example 25. Considering the context in Example 24 and the fuzzy measures

$$\mu_B^{a,0.8}(X) = \begin{cases} 1 & \text{if } X = B \\ 0 & \text{if } X = \varnothing \\ 0.8 & \text{otherwise} \end{cases} \qquad \mu_A^{b,0.8}(Y) = \begin{cases} 1 & \text{if } Y = A \\ 0 & \text{if } Y = \varnothing \\ 0.8 & \text{otherwise} \end{cases}$$

for all $a \in A$ and $b \in B$. The pair $({}^{\uparrow_{\mathrm{IQ}}}, {}^{\downarrow^{\mathrm{IQ}}})$ is a Galois connection and it has three concepts:

$$\begin{aligned} C_0^{\mathrm{IQ}} &= ((b_1/0.4, b_2/1), (a_1/1, a_2/1)) \\ C_1^{\mathrm{IQ}} &= ((b_1/0.7, b_2/1), (a_1/0.4, a_2/1)) \\ C_2^{\mathrm{IQ}} &= ((b_1/1, b_2/1), (a_1/0.4, a_2/0.7)) \end{aligned}$$

where each mapping is represented by an ordered pair. The Hasse diagram of the obtained concept lattice is on the left side of Figure 1.

Notice the difference from the usual concept lattice. Considering the non-quantified concept-forming operators we have the lattice in the right side of Figure 1 and the obtained concepts are the following:

$$\begin{aligned} C_0 &= ((b_1/0.4, b_2/0.8), (a_1/1, a_2/1)) \\ C_1 &= ((b_1/0.4, b_2/1), (a_1/1, a_2/0.8)) \\ C_2 &= ((b_1/0.7, b_2/0.8), (a_1/0.4, a_2/1)) \\ C_3 &= ((b_1/0.7, b_2/1), (a_1/0.4, a_2/0.8)) \\ C_4 &= ((b_1/1, b_2/1), (a_1/0.4, a_2/0.7)) \end{aligned}$$

Thus, these quantifiers provide concept-forming operators similar to the ones considered in the threshold concept lattices, which offer a reduction of the original concept lattice. □

More properties of the new conjunctive and implicative quantified concept-forming operators will be studied in the future.

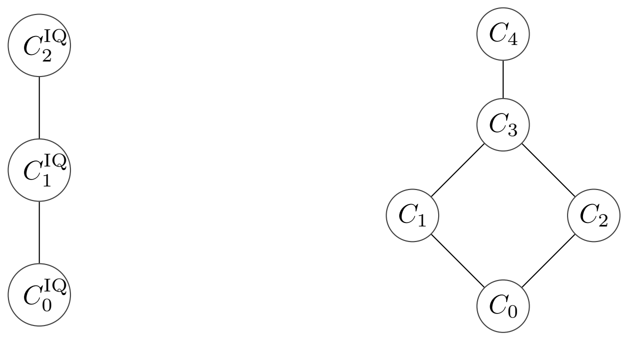

Figure 1: Concept lattices of Example 24.

5. Conclusions and future work

Four generalized quantifiers have been defined, two conjunctive and two implicative ones, which generalize the monadic quantifiers of type $\langle 1 \rangle$ determined by fuzzy measures introduced in [13, 43]. We have proven they also have the universal and existencial quantifiers as particular cases. Moreover, efficient characterizations have been obtained in order to simplify the original definitions. These quantifiers have been applied to obtain quantified concept-forming operators. From the four definitions 16 different possibilities of defining the concept-forming operators exist, this paper has considered one conjunctive and one implicative. However, any combination can be considered and similar properties can be obtained.

We have also proven that the implicative quantified concept-forming operators generalize the operators given in the threshold concept lattices [7, 19, 45], which highlights the interest of these operators and shows a possible interpretation. Furthermore, different examples have been introduced. For instance, we have presented a particular case which absorbs some possible noise in the data removing the smallest value in the computations of the concept-forming operators and so, decreasing the impact of the infimum operator and considering the second smallest value. Finally, we have shown different preliminary properties, which will be increased in the future. Moreover, new definitions will be studied in order to provide Galois connections preserving the flexibility of the generalized quantifiers. In addition, these new operators will be applied to particular real cases, such as in renewable energy and digital forensic datasets.

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