Characterizing reducts in multi-adjoint concept lattices*

M. Eugenia Cornejo[1], Jesús Medina[1], Eloísa Ramírez-Poussa[1]

Department of Mathematics, University of Cádiz. Spain
Email: {mariaeugenia.cornejo,jesus.medina,eloisa.ramirez}@uca.es

Abstract

The construction of reducts, that is, minimal sets of attributes containing the main information of a database, is a fundamental task in different frameworks, such as in Formal Concept Analysis (FCA) and Rough Set Theory (RST). This paper will be focused on a general fuzzy extension of FCA, called multi-adjoint concept lattice, and we present a study about the attributes generating meet-irreducible elements and on the reducts in this framework. From this study, we introduce interesting results on the cardinality of reducts and the consequences in the classical case.

Keywords: formal concept analysis; reducts, attribute reduction

1. Introduction

Nowadays, collected databases generally contain a large amount of data which makes their treatment a really difficult task. In addition, these data usually include redundant information that only serves to increase the complexity to handle the information. Knowledge reduction is a key step in many areas that consider databases, for example, software engineering, information retrieval, data mining, knowledge discovery, machine learning, among others [11, 12, 17, 19, 25, 24].

Formal Concept Analysis [14] is considered a useful tool to treat information contained in databases by using a mathematical structure called

*Partially supported by the Spanish Economy and Competitiveness Ministry (MINECO) project TIN2016-76653-P

**Corresponding author.

concept lattice. These databases are composed by sets of attributes A and objects B related between them by means of a relation $R \subseteq A \times B$. The high complexity in the computation of the concept lattice makes natural to consider mechanisms in order to reduce the set of attributes or objects in the considered database. This important goal in FCA has been studied in diverse papers, which introduce different reduction mechanisms trying to preserve the main information [3, 6, 13, 19, 23, 24, 25, 26].

In [8], the attribute classification theorems based on the categorization of the set of attributes given by Pawlak to RST [22] were presented. These theorems were stated within the fuzzy framework of multi-adjoint concept lattices [20, 21], which is a generalization of FCA that provides a more flexible environment capable of accommodating other fuzzy approaches given in the literature [1, 2, 4, 5, 15, 18]. Specifically, the attribute classification theorems divide into three types the set of attributes -absolutely necessary, relatively necessary and absolutely unnecessary attributes- remaining the process of selection of these attributes for the construction of reducts.

In this paper, our research topic will follow in that direction, going in depth in the study of reducts of any multi-adjoint context. For that purpose, we introduce a new definition related to the attributes that generate meet-irreducible concepts of a multi-adjoint concept lattice. From this notion, we rewrite in a simpler way the attribute classification theorems, making easier their application, and we present several properties relating this notion to the relative necessary attributes. In addition, a study about the cardinality of reducts will be introduced. We will prove that when the set of relatively necessary attributes is non-empty, reducts with different cardinalities arise. In this study, we will show under what conditions the reducts of a multi-adjoint context have the same cardinality. Moreover, we will provide a bound of the cardinality of any reduct, together with more interesting results. The introduced results will be considered in the classical case providing interesting consequences.

Although the paper is focused on the multi-adjoint concept lattice framework, all these results will be essential in order to compute the reducts in other (fuzzy) FCA and RST frameworks. The interesting case of fuzzy RST we will be analyzed in the future.

The paper is organized as follows: an overview associated with preliminary notions of multi-adjoint concept lattice framework and attribute reduction are recalled in Section 2. Section 3 presents several properties about attributes that generate meet-irreducible elements of a concept lattice. This section also includes some results related to the cardinality of reducts together with different examples. The consideration of the previous results in the classical case is given in Section 4. Section 5 finishes with several conclusions and future challenges.

2. Preliminaries

In this section we recall the basic notions and necessary results in order to classify the attributes in the multi-adjoint concept lattice framework.

2.1. Multi-adjoint concept lattice framework

Adjoint triples are the basic computational operators [9, 10] in the considered fuzzy concept lattice framework. These operators are generalizations of a triangular norm (t-norm) and its residuated implication [16].

Definition 1. *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *be posets and* $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ *be mappings, then* $(\&, \swarrow, \nwarrow)$ *is an* adjoint triple *with respect to* P_1, P_2, P_3 *if:*

$$x \leq_1 z \swarrow y \quad \textit{iff} \quad x \,\&\, y \leq_3 z \quad \textit{iff} \quad y \leq_2 z \nwarrow x \tag{1}$$

where $x \in P_1$, $y \in P_2$ *and* $z \in P_3$. *The condition* (1) *is also called* adjoint property.

Once we have recalled this notion, the definitions of multi-adjoint frame and context are given below.

Definition 2. *A* multi-adjoint frame *is a tuple* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *where* $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$ *are complete lattices,* $(P, \leq)$ *is a poset and* $(\&_i, \swarrow^i, \nwarrow_i)$ *is an adjoint triple with respect to* L_1, L_2, P, *for all* $i \in \{1, \dots, n\}$.

Definition 3. *Let* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *be a multi-adjoint frame, a* context *is a tuple* (A, B, R, σ) *such that* A *and* B *are nonempty sets (usually interpreted as attributes and objects, respectively),* R *is a* P*-fuzzy relation* $R\colon A \times B \to P$ *and* $\sigma\colon A \times B \to \{1, \dots, n\}$ *is a mapping which associates any element in* $A \times B$ *with some particular adjoint triple in the frame.*

The concept-forming operators $^\uparrow\colon L_2^B \to L_1^A$ and $^\downarrow\colon L_1^A \to L_2^B$ considered in this framework are defined as

$$g^\uparrow(a) = \inf\{R(a,b) \swarrow^{\sigma(a,b)} g(b) \mid b \in B\} \tag{2}$$

$$f^\downarrow(b) = \inf\{R(a,b) \nwarrow_{\sigma(a,b)} f(a) \mid a \in A\} \tag{3}$$

for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, where L_2^B and L_1^A denote the set of mappings $g\colon B \to L_2$ and $f\colon A \to L_1$, respectively.

These operators form a Galois connection [21] and they are used to compute the corresponding concepts. We say that a pair $\langle g, f\rangle$ with $g \in L_2^B$, $f \in L_1^A$ is a *multi-adjoint concept* if the equalities $g^\uparrow = f$ and $f^\downarrow = g$ hold. The first component of a multi-adjoint concept is a fuzzy subsets of objects g called *extension* and the second component is a fuzzy subsets of attributes f called *intension*.

Definition 4. *The* multi-adjoint concept lattice *associated with a multi-adjoint frame* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *and a context* (A, B, R, σ) *given, is the set*

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^\uparrow = f, f^\downarrow = g\}$$

where the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ *if and only if* $g_1 \preceq_2 g_2$ *(equivalently* $f_2 \preceq_1 f_1$*).*

In [7, 8], we presented a characterization of the *meet-irreducible elements* of a multi-adjoint concept lattice in order to classify the set of attributes of the associated multi-adjoint context. A meet-irreducible element is a concept that cannot be expressed as infimum of strictly greater concepts of the lattice.

Definition 5. *Given a lattice* $(L, \preceq)$*, such that* $\wedge, \vee$ *are the meet and the join operators, and an element* $x \in L$ *verifying*

1. *If* L *has a top element* $\top$*, then* $x \neq \top$*.*
2. *If* $x = y \wedge z$*, then* $x = y$ *or* $x = z$*, for all* $y, z \in L$*.*

we call x meet-irreducible ($\wedge$-irreducible) element *of* L*. Condition* (2) *is equivalent to*

2′. *If* $x < y$ *and* $x < z$*, then* $x < y \wedge z$*, for all* $y, z \in L$*.*

A join-irreducible ($\vee$-irreducible) element *of* L *is defined dually.*

Beside the meet-irreducible elements, the characterization theorem considers the following specific family of fuzzy subsets of attributes.

Definition 6. *For each* $a \in A$*, the fuzzy subsets of attributes* $\phi_{a,x} \in L_1^A$ *defined, for all* $x \in L_1$*, as*

$$\phi_{a,x}(a') = \begin{cases} x & \text{if } a' = a \\ \bot_1 & \text{if } a' \neq a \end{cases}$$

will be called fuzzy-attributes, *where* $\bot_1$ *is the minimum element in* L_1. *The set of all fuzzy-attributes will be denoted as* $\Phi = \{\phi_{a,x} \mid a \in A, x \in L_1\}$.

Theorem 7. *The set of* $\wedge$*-irreducible elements of* $\mathcal{M}$, $M_F(A)$, *is formed by the pairs* $\langle \phi_{a,x}^{\downarrow}, \phi_{a,x}^{\downarrow\uparrow} \rangle$ *in* $\mathcal{M}$, *with* $a \in A$ *and* $x \in L_1$, *such that*

$$\phi_{a,x}^{\downarrow} \neq \bigwedge \{\phi_{a_i,x_i}^{\downarrow} \mid \phi_{a_i,x_i} \in \Phi, \phi_{a,x}^{\downarrow} \prec_2 \phi_{a_i,x_i}^{\downarrow}\}$$

and $\phi_{a,x}^{\downarrow} \neq g_{\top_2}$, *where* $\top_2$ *is the maximum element in* L_2 *and* $g_{\top_2} \colon B \to L_2$ *is the fuzzy subset defined as* $g_{\top_2}(b) = \top_2$, *for all* $b \in B$.

This characterization is essential in order to obtain the attribute classification theorems that will be recalled in the following section.

2.2. Attribute reduction in multi-adjoint contexts

To begin with, we need to introduce the definitions of consistent set and reduct [8].

Definition 8. *A set of attributes* $Y \subseteq A$ *is a* consistent *set of* (A, B, R, σ) *if the following isomorphism holds:*

$$\mathcal{M}(Y, B, R_Y, \sigma_{Y \times B}) \cong_E \mathcal{M}(A, B, R, \sigma)$$

This is equivalent to say that, for all $\langle g, f \rangle \in \mathcal{M}(A, B, R, \sigma)$, *there exists a concept* $\langle g', f' \rangle \in \mathcal{M}(Y, B, R_Y, \sigma_{Y \times B})$ *such that* $g = g'$.

Moreover, if $\mathcal{M}(Y \setminus \{a\}, B, R_{Y \setminus \{a\}}, \sigma_{Y \setminus \{a\} \times B}) \not\cong_E \mathcal{M}(A, B, R, \sigma)$, *for all* $a \in Y$, *then* Y *is called a* reduct *of* (A, B, R, σ).

The core *of* (A, B, R, σ) *is the intersection of all the reducts of* (A, B, R, σ).

The set of attributes can be classified, taking into account the reducts of the associated context.

Definition 9. *Given a formal context* (A, B, R, σ) *and the set* $\mathcal{Y} = \{Y \subseteq A \mid Y \text{ is a reduct}\}$ *of all reducts of* (A, B, R, σ). *The set of attributes* A *can be divided into the following three parts:*

1. *Absolutely necessary attributes (core attribute)* $C_f = \bigcap_{Y \in \mathcal{Y}} Y$.
2. *Relatively necessary attributes* $K_f = (\bigcup_{Y \in \mathcal{Y}} Y) \setminus (\bigcap_{Y \in \mathcal{Y}} Y)$.
3. *Absolutely unnecessary attributes* $I_f = A \setminus (\bigcup_{Y \in \mathcal{Y}} Y)$.

Finally, we include the attribute classification theorems presented in [8].

Theorem 10 ([8]). *Given $a_i \in A$, we have that $a_i \in C_f$ if and only if there exists $x_i \in L_1$, such that $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \in M_F(A)$, satisfying that $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \neq \langle \phi^{\downarrow}_{a_j,x_j}, \phi^{\downarrow\uparrow}_{a_j,x_j} \rangle$, for all $x_j \in L_1$ and $a_j \in A$, with $a_j \neq a_i$.*

Theorem 11 ([8]). *Given $a_i \in A$, we have that $a_i \in K_f$ if and only if $a_i \notin C_f$ and there exists $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \in M_F(A)$ satisfying that E_{a_i,x_i} is not empty and $A \setminus E_{a_i,x_i}$ is a consistent set, where the sets $E_{a_i,x}$ with $a_i \in A$ and $x \in L_1$ are defined as:*

$$E_{a_i,x} = \{a_j \in A \setminus \{a_i\} \mid \textit{there exist } x' \in L_1, \textit{ satisfying } \phi^{\downarrow}_{a_i,x} = \phi^{\downarrow}_{a_j,x'}\}$$

Theorem 12 ([8]). *Given $a_i \in A$, it is absolutely unnecessary, $a_i \in I_f$, if and only if, for each $x_i \in L_1$, we have that $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \notin M_F(A)$, or in the case that $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \in M_F(A)$, then $A \setminus E_{a_i,x_i}$ is not a consistent set.*

The previous results allow us to give a classification of the set of attributes in absolutely necessary, relatively necessary and absolutely unnecessary attributes. From this classification we can obtain reducts, which may significantly reduce the computational complexity of the concept lattice.

3. Reducts of a multi-adjoint context

Besides knowing how the set of attributes is classified, it is necessary to study how this classification influences in the selection of attributes to build reducts. In this section, we will analyze the construction process of reducts from the attribute classification shown in the previous section.

Evidently, the absolutely unnecessary attributes must be directly removed. In addition, the absolutely necessary attributes, the attributes in the core, must be included in all the reducts. Therefore, the main task is the selection of the relatively necessary attributes, since when the set of relatively necessary attributes is nonempty, several reducts are obtained. Before starting this study, we will present some of the questions that we want to answer: Do all the reducts have the same cardinality? What is the size of reducts? How can we bound the size of reducts? When have reducts got the same cardinality? How many minimal reducts has a context?

3.1. Attributes generating meet-irreducible elements

The answers to these previous questions will be based on the properties of the subset of attributes generating a given concept.

Definition 13. *Given a multi-adjoint frame* $(L_1, L_2, P, \&_1, \dots, \&_n)$*, a context* (A, B, R, σ) *associated with the concept lattice* $(\mathcal{M}, \preceq)$ *and a concept* C *of* $(\mathcal{M}, \preceq)$*,* the set of attributes generating C *is defined as the set:*

$$Atg(C) = \{a \in A \mid \textit{there exists} \quad \phi_{a,x} \in \Phi \quad \textit{such that} \quad \langle \phi^{\downarrow}_{a,x}, \phi^{\downarrow\uparrow}_{a,x} \rangle = C\}$$

First of all, we show that these sets are nonempty for meet-irreducible concepts.

Proposition 14. *If* C *is a meet-irreducible concept of* $(\mathcal{M}, \preceq)$*, then* $Atg(C)$ *is a nonempty set.*

Proof: By Theorem 7, if $C \in (\mathcal{M}, \preceq)$ is a meet-irreducible concept, then $C = \langle \phi^{\downarrow}_{a,x}, \phi^{\downarrow\uparrow}_{a,x} \rangle$ satisfying that $\phi^{\downarrow}_{a,x} \neq \bigwedge \phi^{\downarrow}_{a_i,x_i}$ where $\phi^{\downarrow}_{a,x} \prec_2 \phi^{\downarrow}_{a_i,x_i}$ and $\phi^{\downarrow}_{a,x} \neq g_{\top_2}$. Therefore, in particular, we can conclude that $a \in \mathrm{Atg}(C)$ and consequently $\mathrm{Atg}(C) \neq \varnothing$. □

The following example presents a particular context which will be considered for illustrating the results in this section.

Example 15. *Let* $(L, \preceq, \&_G)$ *be a multi-adjoint frame, where* $\&_G$ *is the Gödel conjunctor with respect to* $L = \{0, 0.5, 1\}$*. In this framework, the context is* (A, B, R, σ)*, where* $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$*,* $B = \{b_1, b_2, b_3\}$*,* $R\colon A \times B \to L$ *is given by the table in Figure 1, and* σ *is constant.*

The concept lattice of the considered framework and context are displayed in Figure 1, from which it is easy to see that the meet-irreducible elements are C_0*,* C_1*,* C_2 *and* C_3*. Now, we will show that the sets* $Atg(C_0)$*,* $Atg(C_1)$*,* $Atg(C_2)$ *and* $Atg(C_3)$ *are not empty. For that, the fuzzy-attributes associated with the meet-irreducible concepts need to be obtained. Applying the concept-forming operators to the fuzzy-attributes we have*

$$\begin{array}{rcl}
\langle \phi^{\downarrow}_{a_6,1.0}, \phi^{\downarrow\uparrow}_{a_6,1.0} \rangle & = & C_0 \\
\langle \phi^{\downarrow}_{a_2,1.0}, \phi^{\downarrow\uparrow}_{a_2,1.0} \rangle = \langle \phi^{\downarrow}_{a_3,1.0}, \phi^{\downarrow\uparrow}_{a_3,1.0} \rangle & = & C_1 \\
\langle \phi^{\downarrow}_{a_1,0.5}, \phi^{\downarrow\uparrow}_{a_1,0.5} \rangle = \langle \phi^{\downarrow}_{a_1,1.0}, \phi^{\downarrow\uparrow}_{a_1,1.0} \rangle & = & \\
\langle \phi^{\downarrow}_{a_2,0.5}, \phi^{\downarrow\uparrow}_{a_2,0.5} \rangle = \langle \phi^{\downarrow}_{a_3,0.5}, \phi^{\downarrow\uparrow}_{a_3,0.5} \rangle = \langle \phi^{\downarrow}_{a_6,0.5}, \phi^{\downarrow\uparrow}_{a_6,0.5} \rangle & = & C_2 \\
\langle \phi^{\downarrow}_{a_4,1.0}, \phi^{\downarrow\uparrow}_{a_4,1.0} \rangle & = & C_3
\end{array}$$

obtaining the association which is written in Table 1.

R	b_1	b_2	b_3
a_1	1	1	0
a_2	0.5	1	0
a_3	0.5	1	0
a_4	1	1	0.5
a_5	1	1	1
a_6	0.5	0.5	0

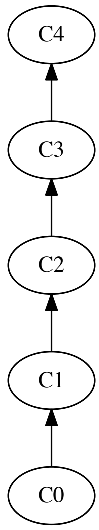

Figure 1: Relation R and Hasse diagram of Example 15.

$M_F(A)$	Fuzzy-attributes generating the meet-irreducible concept
C_0	$\phi_{a_6,1.0}$
C_1	$\phi_{a_2,1.0}, \phi_{a_3,1.0}$
C_2	$\phi_{a_1,0.5}, \phi_{a_1,1.0}, \phi_{a_2,0.5}, \phi_{a_3,0.5}, \phi_{a_6,0.5}$
C_3	$\phi_{a_4,1.0}$

Table 1: Fuzzy-attributes generating the meet-irreducible concepts of Example 15.

From this table, the sets of attributes generating these concepts are straightforwardly determined:

$$\begin{array}{rcl} Atg(C_0) & = & \{a_6\} \\ Atg(C_1) & = & \{a_2, a_3\} \\ Atg(C_2) & = & \{a_1, a_2, a_3, a_6\} \\ Atg(C_3) & = & \{a_4\} \end{array}$$

Hence, these subsets of attributes are nonempty as Proposition 14 shows.□

In the following we will show the adaptation of the attribute classification theorems introduced in [8], based on Definition 13. These results are interesting because they simplify the original theorems, by using a simpler language and notation, which make them easier to understand and apply. Therefore, we can obtain the final classification in a more easy and intuitive way. The demonstrations of Theorems 16, 17 and 19 will be omitted since they follow analogously to the ones given in [8].

The first one shows the classification of the absolutely necessary attributes, based on Definition 13.

Theorem 16. *Given an attribute $a \in A$, then $a \in C_f$ if and only if there exists a meet-irreducible concept C of $(\mathcal{M}, \preceq)$ satisfying that $a \in Atg(C)$ and $card(Atg(C)) = 1$.*

The characterization of the relatively necessary attributes is given below.

Theorem 17. *Given an attribute $a \in A$, then $a \in K_f$ if and only if $a \notin C_f$ and there exists $C \in M_F(A)$ with $a \in Atg(C)$ and $card(Atg(C)) > 1$, satisfying that $\big(A \setminus Atg(C)\big) \bigcup \{a\}$ is a consistent set.*

The following result is a consequence of the previous one. It will be useful in the proof of one of the most important results in Section 3.2.

Corollary 18. *Given an attribute $a \in K_f$, there exists a concept $C \in M_F(A)$ such that $a \in Atg(C)$ satisfying that $card(Atg(C) \bigcap K_f) \geq 2$.*

Finally, the adaptation of the theorem that classifies the absolutely unnecessary attributes is shown in the next proposition.

Theorem 19. *Given an attribute $a \in A$, then $a \in I_f$ if and only if, for any $C \in M_F(A)$, $a \notin Atg(C)$, or if $a \in Atg(C)$ then $\big(A \setminus Atg(C)\big) \bigcup \{a\}$ is not a consistent set.*

As we mentioned above, this new version of the attribute classification theorems provides a simpler way to obtain the final classification.

From Example 15 and considering the classification theorems we obtain the following attribute classification:

$$\begin{aligned} I_f &= \{a_1, a_5\} \\ K_f &= \{a_2, a_3\} \\ C_f &= \{a_4, a_6\} \end{aligned}$$

Moreover, from Theorem 16, the singleton sets of attributes generating a concept are characterized by the absolutely necessary attributes. For instance, in Example 15, we have that Atg(C_0) and Atg(C_3) are the only singleton sets (where C_0, C_3 are meet-irreducible concepts) and they are only composed of the attributes a_6 and a_4, respectively, which are the only attributes in the core.

Relatively necessary attributes play an important role in the computation of the reducts. Theorem 17 provides an interesting property of these attributes and the sets $\mathrm{Atg}(C)$ containing these attributes but more properties are needed. The first one deals with the set of attributes generating a meet-irreducible element, when the intersection with the set of relatively necessary attributes is not empty.

Proposition 20. *If C is a meet-irreducible concept of $(\mathcal{M}, \preceq)$ and $Atg(C) \cap K_f \neq \varnothing$ then $card(Atg(C)) \geq 2$.*

Proof: If we assume that $\mathrm{card}(\mathrm{Atg}(C)) = 1$, then by hypothesis we have that $\mathrm{Atg}(C) \cap K_f = \{a\}$ and $a \in K_f$, which leads us to a contradiction with Theorem 16. Hence, the inequality $\mathrm{card}(\mathrm{Atg}(C)) \geq 2$ holds. □

Example 21. *Considering again Example 15, the concept C_2 is a meet-irreducible element such that $Atg(C_2) \cap K_f = \{a_1, a_2, a_3, a_6\} \cap \{a_2, a_3\} = \{a_2, a_3\} \neq \varnothing$. As a consequence, we have that $card(Atg(C_2)) = 4 > 2$ as Proposition 20 shows. In a similar situation the concept C_1 is.* □

The following results show that, if C is a meet-irreducible concept and $\mathrm{Atg}(C)$ does not contain an attribute in the core, then this concept is obtained from at least two different relatively necessary attributes.

Proposition 22. *Let C be a meet-irreducible concept. If $Atg(C) \cap C_f = \varnothing$ then $card(Atg(C) \cap K_f) \geq 2$.*

Proof: If $\mathrm{Atg}(C) \cap C_f = \varnothing$ then, for all $a \in \mathrm{Atg}(C)$ we have that $a \notin C_f$. In addition, since $C \in M_F(A)$ then, by Theorem 17, there exist at least two attributes $a_i, a_j \in K_f$ such that $a_i, a_j \in \mathrm{Atg}(C)$. Consequently, $\mathrm{card}(\mathrm{Atg}(C) \cap K_f) \geq 2$. □

Observe that the converse is no true, in general. We can find a meet-irreducible concept C, satisfying that $\mathrm{card}(\mathrm{Atg}(C) \cap K_f) \geq 2$ and $\mathrm{Atg}(C) \cap C_f \neq \varnothing$. This can be seen when we take into account the concept C_2 in Example 15. Now, $\mathrm{Atg}(C_2) = \{a_1, a_2, a_3, a_6\}$. Therefore, $\mathrm{Atg}(C_2) \cap C_f = \{a_6\}$ and $\mathrm{Atg}(C_2) \cap K_f = \{a_2, a_3\}$.

The next proposition guarantees that, at least one meet-irreducible element satisfying the conditions shown in the previous proposition exists, if the set of relatively necessary attributes is not empty. This result also enriches Corollary 18.

Proposition 23. *If K_f is not empty, then there exists a meet-irreducible concept C, such that $Atg(C) \cap C_f = \varnothing$ and $card(Atg(C) \cap K_f) \geq 2$.*

Proof: If $K_f \neq \varnothing$, then, by Theorem 17, there exists $C \in M_F(A)$ with $a \in \mathrm{Atg}(C)$ and $\mathrm{card}(\mathrm{Atg}(C)) > 1$, satisfying that $\big(A \setminus \mathrm{Atg}(C)\big) \bigcup \{a\}$ is a consistent set. Therefore, $\mathrm{Atg}(C) \cap C_f = \varnothing$, since otherwise in the computation of the set $\big(A \setminus \mathrm{Atg}(C)\big)$ we remove at least an attribute in C_f and the obtained set, $\big(A \setminus \mathrm{Atg}(C)\big) \bigcup \{a\}$ is consistent, which contradicts that all the attributes in the core C_f are in all the reducts. Moreover, by Proposition 22, we can conclude that $\mathrm{card}(\mathrm{Atg}(C) \cap K_f) \geq 2$. $\square$

For example, in Example 15, $K_f = \{a_2, a_3\} \neq \varnothing$ and we have that the meet-irreducible concept C_1 satisfies $\mathrm{Atg}(C_1) \cap C_f = \varnothing$ and $\mathrm{Atg}(C_1) \cap K_f = \{a_2, a_3\}$, where $\mathrm{Atg}(C_1) = \{a_2, a_3\}$ and $C_f = \{a_4, a_6\}$.

3.2. On the cardinality of reducts

In this section, we will take into account the previous results in order to show several statements about the cardinality of the reducts of a multi-adjoint context.

The next result anticipates the sufficient condition we will use to ensure that the cardinality of all the reducts coincides.

Proposition 24. *If $\mathcal{G}_K = \{Atg(C) \mid C \in M_F(A) \quad and \quad Atg(C) \cap K_f \neq \varnothing\}$ is a partition of K_f, each attribute in K_f generates only one meet-irreducible element of the concept lattice.*

Proof: Given $a \in K_f$, by Theorem 17, we have that there exists a concept $C \in M_F(A)$ such that $a \in \mathrm{Atg}(C)$. If we can find another $\wedge$-irreducible concept, C', satisfying that $a \in \mathrm{Atg}(C')$, we obtain that $\mathrm{Atg}(C) \bigcap \mathrm{Atg}(C') \neq \varnothing$ which is a contradiction since $\mathcal{G}_K$ is a partition of K_f. Consequently, the attribute a only generates one meet-irreducible element. $\square$

If $\mathcal{G}_K$ is a partition of K_f we can guarantee that all the reducts have the same cardinality, as the following result proves.

Theorem 25. *When the set*

$$\mathcal{G}_K = \{Atg(C) \mid C \in M_F(A) \quad and \quad Atg(C) \cap K_f \neq \varnothing\}$$

is a partition of K_f*, then:*

(a) All the reducts $Y \subseteq A$ *have the same cardinality and, specifically, the cardinality is:*

$$card(Y) = card(C_f) + card(\mathcal{G}_K)$$

(b) The number of different reducts obtained from the multi-adjoint context is

$$\prod_{Atg(C)\in\mathcal{G}_K} card(Atg(C))$$

Proof: First of all, we will prove item (a). Given any reduct Y of the context (A, B, R, σ), we consider the set of attributes $Y' = Y \setminus C_f$ and the set $C_{K_f} = \{C \in M_F(A) \mid \mathrm{Atg}(C) \cap K_f \neq \varnothing\}$. In order to prove that all reducts have the same cardinality, we define a mapping

$$\begin{aligned} F\colon Y' &\longrightarrow C_{K_f} \\ a &\longmapsto C \end{aligned}$$

which associates each attribute a in Y' with a meet-irreducible concept C of C_{K_f} such that $a \in \mathrm{Atg}(C)$ and we demonstrate that F is a bijection. This mapping is well-defined since if $a \in Y'$ then, $a \in K_f$ and, by Corollary 18, there exists a meet-irreducible concept C such that $a \in \mathrm{Atg}(C)$. Therefore, $F(a) = C \in C_{K_f}$.

In order to show that F is order-embedding, we will consider two attributes $a_1, a_2 \in Y'$ such that $F(a_1) = F(a_2)$ and we will prove by reductio ad absurdum that $a_1 = a_2$. Hence, we will assume that $a_1 \neq a_2$ we will obtain a contradiction. As $F(a_1) = F(a_2) = C$ then $a_1, a_2 \in \mathrm{Atg}(C)$ with $C \in M_F(A)$ and $\mathrm{Atg}(C) \cap K_f \neq \varnothing$. Since $\mathcal{G}_K$ is a partition of K_f, by Proposition 24, we obtain that each attribute $a \in K_f$ generates only one meet-irreducible concept, therefore we can remove the attribute a_1 from the set Y' and, as a consequence, we obtain that $Y \setminus \{a_1\} = Y' \setminus \{a_1\} \cup C_f$ is a consistent set, which contradicts that Y is a reduct. Hence, we can conclude that $a_1 = a_2$.

Now, we will prove that F is onto. If $C \in C_{K_f}$ then $C \in M_F(A)$ and $\mathrm{Atg}(C) \cap K_f \neq \varnothing$. On the other hand, $Y = Y' \cup C_f$ is a reduct and therefore, there exists $a \in Y$ with $a \in \mathrm{Atg}(C)$. Due to the set $\mathcal{G}_K$ is a partition of K_f, $\mathrm{Atg}(C) \cap K_f \neq \varnothing$ and Proposition 24, we can ensure that $\mathrm{Atg}(C) \cap C_f = \varnothing$, hence $a \in K_f$ and, consequently, $a \in Y'$ satisfying $F(a) = C$.

Due to the mapping F is a bijection, we can assert that $\mathrm{card}(Y') = \mathrm{card}(C_{K_f})$. Taking into account that $Y' = Y \setminus C_f$, the last equality can be rewritten as $\mathrm{card}(Y \setminus C_f) = \mathrm{card}(C_{K_f})$, or equivalently, $\mathrm{card}(Y) - \mathrm{card}(Y \cap C_f) = \mathrm{card}(C_{K_f})$. As a consequence, since $Y \cap C_f = C_f$, we obtain that $\mathrm{card}(Y) = \mathrm{card}(C_f) + \mathrm{card}(C_{K_f})$.

It only remains to prove that $\mathrm{card}(C_{K_f}) = \mathrm{card}(\mathcal{G}_K)$, but this equality is straightforwardly obtained because a concept C is in C_{K_f} if and only if the set $\mathrm{Atg}(C)$ belongs to $\mathcal{G}_K$. Therefore, C_{K_f} and $\mathcal{G}_K$ have the same number of elements.

Concerning item (b), we need to consider a subset of attributes from which we can obtain each $\wedge$-irreducible element of the concept lattice. For any reduct $Y \subseteq A$, since $C_f \subseteq Y$, and the set

$$\mathcal{G}_K = \{\mathrm{Atg}(C) \mid C \in M_F(A) \quad \text{and} \quad \mathrm{Atg}(C) \cap K_f \neq \varnothing\}$$

is a partition of K_f, we have to choose one attribute for each element in $\mathcal{G}_K$. Considering the multiplication principle, we deduce that all possible combinations to select these attributes are

$$\prod_{\mathrm{Atg}(C) \in \mathcal{G}_K} \mathrm{card}(\mathrm{Atg}(C))$$

□

The next example clarifies the results stated in the previous theorem.

Example 26. *Let $(L_1, L_2, L_3, \preceq, \&_G^*)$ be the multi-adjoint frame composed by regular partitions of the unit interval in 10, 4 and 5 pieces, that is, $L_1 = [0,1]_{10}$, $L_2 = [0,1]_4$, $L_3 = [0,1]_5$, respectively, and the discretization of Gödel conjunctor $\&_G^* \colon L_1 \times L_2 \to L_3$. The considered context (A, B, R, σ) is composed by the set of attributes $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, the set of objects $B = \{b_1, b_2, b_3\}$, the relation $R \colon A \times B \to L_3$ displayed in the left side of Figure 2 and the mapping σ which is constantly $\&_G^*$.*

From this framework and this context, we obtain the Hasse diagram of the concept lattice displayed in the right side of Figure 2. Considering the Hasse diagram, we obtain the set of meet-irreducible concepts $M_F(A) = \{C_1, C_8, C_9, C_{10}, C_{13}, C_{14}\}$. Table 2 shows the fuzzy-attributes associated with these concepts.

R	a_1	a_2	a_3	a_4	a_5	a_6
b_1	0.6	0.2	0.2	0	1	0.6
b_2	0.8	0.4	0.6	0.6	1	0.8
b_3	0.6	0.6	0.2	0	0	0

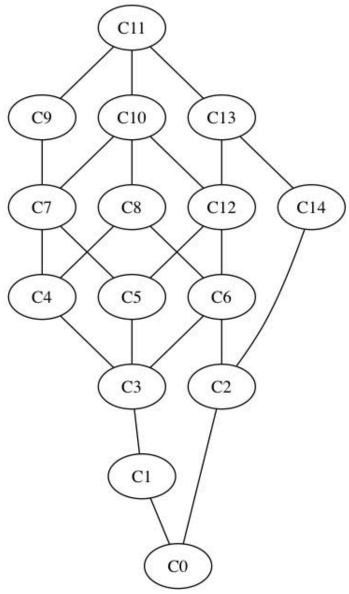

Figure 2: Relation R (left side) and Hasse diagram of $(\mathcal{M}, \preceq)$ (right side) of Example 26.

Moreover, the sets of attributes generating each meet-irreducible element of the concept lattice, which are obtained by Definition 13, are listed below:

$$\begin{aligned} Atg(C_1) &= \{a_3, a_4\} \\ Atg(C_8) &= \{a_1\} \\ Atg(C_9) &= \{a_5, a_6\} \\ Atg(C_{10}) &= \{a_1\} \\ Atg(C_{13}) &= \{a_2\} \\ Atg(C_{14}) &= \{a_2\} \end{aligned}$$

Taking into account these previous sets and applying the adaptation of the attribute classification theorems we have that:

$$\begin{aligned} C_f &= \{a_1, a_2\} \\ K_f &= \{a_3, a_4, a_5, a_6\} \end{aligned}$$

Our goal is to compute all possible reducts. First of all, it is important

$M_F(A)$	Fuzzy-attributes generating the meet-irreducible concept
C_1	$\phi_{a_3,0.7}, \phi_{a_3,0.8}, \phi_{a_3,0.9}, \phi_{a_3,1.0}$ $\phi_{a_4,0.7}, \phi_{a_4,0.8}, \phi_{a_4,0.9}, \phi_{a_4,1.0}$
C_8	$\phi_{a_1,0.9}, \phi_{a_1,1.0}$
C_9	$\phi_{a_5,0.1}, \phi_{a_5,0.2}, \phi_{a_5,0.3}, \phi_{a_5,0.4}, \phi_{a_5,0.5}, \phi_{a_5,0.6}, \phi_{a_5,0.7}, \phi_{a_5,0.8}, \phi_{a_5,0.9}, \phi_{a_5,1.0}$ $\phi_{a_6,0.1}, \phi_{a_6,0.2}, \phi_{a_6,0.3}, \phi_{a_6,0.4}, \phi_{a_6,0.5}, \phi_{a_6,0.6}$
C_{10}	$\phi_{a_1,0.7}, \phi_{a_1,0.8}$
C_{13}	$\phi_{a_2,0.3}, \phi_{a_2,0.4}$
C_{14}	$\phi_{a_2,0.5}, \phi_{a_2,0.6}$

Table 2: Fuzzy-attributes generating the meet-irreducible concepts of Example 26.

to note that the attributes a_1 and a_2 will be included in all reducts since they belong to the core attribute. Now, we will focus on the selection of the relatively necessary attributes that should be contained in each reduct. With this purpose, we compute the following set:

$$\begin{aligned} \mathcal{G}_K &= \{Atg(C) \mid C \in M_F(A) \quad and \quad Atg(C) \cap K_f \neq \varnothing\} \\ &= \{Atg(C_1), Atg(C_9)\} \\ &= \{\{a_3, a_4\}, \{a_5, a_6\}\} \end{aligned}$$

It is easy to see that $\mathcal{G}_K$ is a partition of K_f due to $Atg(C_1)$ and $Atg(C_9)$ are disjoint subsets of K_f. This fact allows to apply Proposition 24 and Theorem 25 to this multi-adjoint context obtaining that:

(1) Each relatively necessary attribute only generates one meet-irreducible element of the concept lattice. According Table 2, we obtain that the meet-irreducible concept C_1 is exclusively generated by a_3 and a_4. In addition, the attributes a_5 and a_6 only generate the concept C_9.

(2) Each reduct Y of this context satisfies that $card(Y) = card(C_f) + card(\mathcal{G}_K) = 2 + 2 = 4$. In other words, all reducts have the same cardinality. From this context, we obtain

$$\prod_{Atg(C) \in \mathcal{G}_K} card(Atg(C)) = 2 \cdot 2 = 4$$

different reducts which are shown below:

$$\begin{array}{rcl} Y_1 &=& \{a_1, a_2, a_3, a_5\} \\ Y_2 &=& \{a_1, a_2, a_3, a_6\} \\ Y_3 &=& \{a_1, a_2, a_4, a_5\} \\ Y_4 &=& \{a_1, a_2, a_4, a_6\} \end{array}$$

These previous reducts provide the next isomorphic concept lattices:

$$(\mathcal{M}, \preceq) \cong (\mathcal{M}^{Y_1}, \preceq) \cong (\mathcal{M}^{Y_2}, \preceq) \cong (\mathcal{M}^{Y_3}, \preceq) \cong (\mathcal{M}^{Y_4}, \preceq)$$

□

In Theorem 25 we have provided a sufficient condition in order to ensure that the cardinality of the reducts is the same. Example 27 reveals that it is not a necessary condition.

Example 27. *From Example 15, we can ensure that either attribute a_2 or a_3 is needed (the attribute a_1 is absolutely unnecessary) in order to obtain the meet-irreducible concept C_1. Hence, since $a_4, a_6 \in C_f$, two reducts $Y_1 = \{a_2, a_4, a_6\}$ and $Y_2 = \{a_3, a_4, a_6\}$ exist. Thus, only three attributes are needed in order to consider a concept lattice isomorphic to the original one.*

Since the set $\mathcal{G}_K$ is composed by the attributes generating C_1 and C_2, we have that $\mathcal{G}_K = \{\{a_2, a_3\}, \{a_1, a_2, a_3, a_6\}\}$. Then, we can see clearly that $\mathcal{G}_K$ is not a partition of K_f although, in this case, all the reducts have the same cardinality. □

The following example shows a particular situation where the obtained reducts have different cardinality and the set $\mathcal{G}_K$ is not a partition of K_f.

Example 28. *Given the same framework that in Example 26, we will consider a context composed by seven attributes, three objects and the relation R included in Table 3. Due to this relation is very similar to the relation of Example 26, the obtained concept lattice is isomorphic to the one displayed in Figure 2.*

According to the sets of attributes generating each meet-irreducible element of the concept lattice:

$$\begin{array}{rcl} Atg(C_1) &=& \{a_6, a_7\} \\ Atg(C_8) &=& \{a_1\} \\ Atg(C_9) &=& \{a_4, a_5, a_6\} \\ Atg(C_{10}) &=& \{a_1\} \\ Atg(C_{13}) &=& \{a_2\} \\ Atg(C_{14}) &=& \{a_2\} \end{array}$$

Table 3: Definition of R

R	a_1	a_2	a_3	a_4	a_5	a_6	a_7
b_1	0.6	0.2	0.2	1	0.6	0.2	0
b_2	0.8	0.4	0.4	1	0.8	0.6	0.6
b_3	0.6	0.6	0.2	0	0	0.2	0

and considering the attribute classification shown in Propositions 16, 17 and 19, we obtain that:

$$\begin{aligned} C_f &= \{a_1, a_2\} \\ K_f &= \{a_4, a_5, a_6, a_7\} \\ I_f &= \{a_3\} \end{aligned}$$

From the previous classification, it is easy to see that attributes a_1 and a_2 will be contained in all reducts and the attribute a_3 will be not considered in the construction process of reducts.

In order to obtain the whole set of meet-irreducible concepts by means of reducts, we need to choose an attribute of $Atg(C_1)$ and another one of $Atg(C_9)$. Note that, the set $\mathcal{G}_K = \{Atg(C_1), Atg(C_9)\}$ is not a partition of K_f, since $Atg(C_1) \cap Atg(C_9) = \{a_6\} \neq \varnothing$. As a consequence, the size of reducts will depend on the selected attributes. The obtained reducts are listed below:

$$\begin{aligned} Y_1 &= \{a_1, a_2, a_6\} \\ Y_2 &= \{a_1, a_2, a_4, a_7\} \\ Y_3 &= \{a_1, a_2, a_5, a_7\} \end{aligned}$$

□

Now, we will provide a lower bound and an upper bound of the cardinality of every reduct.

Proposition 29. *Given $\mathcal{G}_K^C = \{Atg(C) \mid C \in M_F(A) \quad \text{such that} \quad Atg(C) \cap K_f \neq \varnothing \quad \text{and} \quad Atg(C) \cap C_f = \varnothing\}$ and any reduct Y of the context (A, B, R, σ). The cardinality of the set Y can be bounded as follows:*

$$card(C_f) \leq card(Y) \leq card(C_f) + card(\mathcal{G}_K^C)$$

Proof: The first inequality holds because of Y is a reduct of the context (A, B, R, σ) and, therefore, $C_f \subseteq Y$.

In order to obtain the second inequality, we consider the set $M_F(A) = \{C_1, \ldots, C_n\}$ and we define the mapping:

$$H \colon Y \setminus C_f \to \mathcal{G}_K^C$$

as $H(a) = \mathrm{Atg}(C_i)$ such that $C_i \in M_F(A)$, $a \in \mathrm{Atg}(C_i)$, $\mathrm{Atg}(C_i) \bigcap Y = \{a\}$ and if more than one concept satisfying the previous conditions exist, we will consider the concept with the smallest subindex. Now, we will prove that H is well defined, that is, $H(a)$ is an element in $\mathcal{G}_K^C$. Given $a \in Y \setminus C_f$, since Y is a reduct and $a \notin C_f$, we have that $a \in K_f$. Consequently, by Proposition 23, there exists at least a $\wedge$-irreducible element $C \in M_F(A)$ such that $a \in \mathrm{Atg}(C)$. In addition, if $\{a\} \subsetneq \mathrm{Atg}(C) \bigcap Y$ for each $C \in M_F(A)$ with $a \in \mathrm{Atg}(C)$, then we can conclude that the set $Y \setminus \{a\}$ is a consistent set, which is a contradiction since Y is a reduct. Therefore, we can always find a concept $C \in M_F(A)$ such that $a \in \mathrm{Atg}(C)$ and $\{a\} = \mathrm{Atg}(C) \bigcap Y$. Hence, since only one is considered when more than one concept satisfy the previous conditions, we can ensure that this mapping is well defined.

Furthermore, if $a \neq a'$ then $H(a) \neq H(a')$, because of if $H(a) = H(a')$, then $\mathrm{Atg}(C) \bigcap Y = \{a, a'\}$ which is a contradiction. As a consequence, H is an injective function and we can conclude that $\mathrm{card}(Y \setminus C_f) \leq \mathrm{card}(\mathcal{G}_K^C)$ or, equivalently, $\mathrm{card}(Y) \leq \mathrm{card}(C_f) + \mathrm{card}(\mathcal{G}_K^C)$. $\square$

From the last proposition, the following corollary is obtained.

Corollary 30. *Given* $\mathcal{G}_K = \{Atg(C) \mid C \in M_F(A) \quad and \quad Atg(C) \cap K_f \neq \varnothing\}$ *and any reduct* Y *of the context* (A, B, R, σ)*. Then, the following chain is always satisfied:*

$$card(C_f) \leq card(Y) \leq card(C_f) + card(\mathcal{G}_K)$$

Observe that, since $\mathcal{G}_K^C \subseteq \mathcal{G}_K$ the upper approximation given in Proposition 29 is better than the one given in Corollary 30, in general.

In addition, when $\mathcal{G}_K$ is a partition of K_f then each $\mathrm{Atg}(C) \in \mathcal{G}_K$ satisfies, by Proposition 24, that $\mathrm{Atg}(C) \cap C_f = \varnothing$. Therefore, if $\mathcal{G}_K$ is a partition of K_f, the equality $\mathcal{G}_K^C = \mathcal{G}_K$ holds, which shows in this case that the approximations given in Proposition 29 and Corollary 30 are equals.

Example 31. *Returning to Example 15, we will see that both reducts Y_1 and Y_2 satisfy the inequalities in Proposition 29 and Corollary 30. Taking into account the definitions of the sets $\mathcal{G}_K$ and $\mathcal{G}_K^C$, we obtain that:*

$$\begin{array}{rcl} \mathcal{G}_K & = & \{Atg(C_1), Atg(C_2)\} = \{\{a_2, a_3\}, \{a_1, a_2, a_3, a_6\}\} \\ \mathcal{G}_K^C & = & \{Atg(C_1)\} = \{\{a_2, a_3\}\} \end{array}$$

Considering the reducts $Y_1 = \{a_2, a_4, a_6\}$ and $Y_2 = \{a_3, a_4, a_6\}$, the following chains of inequalities hold:

$$2 = card(C_f) \leq card(Y_1) = card(Y_2) \leq card(C_f) + card(\mathcal{G}_K^C) = 3$$

$$2 = card(C_f) \leq card(Y_1) = card(Y_2) \leq card(C_f) + card(\mathcal{G}_K) = 4$$

Clearly, the set $\mathcal{G}_K^C$ provides a more accurate upper bound with respect to the cardinality of the reducts than the set $\mathcal{G}_K$, as we mentioned above. □

We have considered the sets $\mathcal{G}_K$ and $\mathcal{G}_K^C$ above, in which the unnecessary attributes are included. We will finish this section presenting an interesting property arising when the attributes of I_f are removed from $\mathcal{G}_K^C$. In order to present this property, we will define the following set:

$$\mathcal{G}_K^{C,I} = \{\text{Atg}(C) \setminus I_f \mid \text{Atg}(C) \in \mathcal{G}_K^C\}$$

From now on, the elements belonging to $\mathcal{G}_K^{C,I}$ will be denoted as $\text{Atg}(C)^K$. Obviously, the equality $\text{card}\{\mathcal{G}_K^C\} = \text{card}\{\mathcal{G}_K^{C,I}\}$ holds. The following result shows a specific case in which we can easily determine the cardinality of the minimal reducts, and the number of different minimal reducts of a multi-adjoint context.

Proposition 32. *If $\bigcap \mathcal{G}_K^{C,I} \neq \varnothing$ then there exists a reduct Y such that $card(Y) = card(C_f) + 1$ and, therefore, Y is a minimal reduct.*

Moreover, the number of minimal reducts of the multi-adjoint context is $card\{\bigcap \mathcal{G}_K^{C,I}\}$.

Proof: If $\bigcap \mathcal{G}_K^{C,I} \neq \varnothing$ then, there exists $a_0 \in \bigcap \mathcal{G}_K^{C,I}$ such that $a_0 \in K_f$ and $a_0 \in \text{Atg}(C)$ for all $\text{Atg}(C) \in \mathcal{G}_K^C$. Considering the subset $Y = C_f \bigcup \{a_0\}$, we obtain all $\wedge$-irreducible elements of the concept lattice and so, an isomorphic concept lattice to the original one. Consequently, the subset Y is a consistent set of the context. In addition, let us prove that Y is a reduct.

Since, by definition, we cannot remove any attribute in C_f, we focus the attention on a_0. Due to $a_0 \in K_f$, by Proposition 23, there exists a meet-irreducible concept C, such that $a_0 \in \text{Atg}(C)$, $\text{Atg}(C) \cap C_f = \varnothing$ and $\text{Atg}(C) \cap K_f \neq \varnothing$. Therefore, by the definition of Y, the concept C is not generated by other attribute different from a_0 and so, the set $Y \setminus \{a_0\}$ is not consistent. Consequently, the subset Y is a reduct and $\text{card}(Y) = \text{card}(C_f) + 1$.

On the other hand, the reduct Y is minimal since we cannot build a reduct considering a smaller number of attributes. Moreover, the number of different minimal reducts that we can build in this way will be $\text{card}\{\bigcap \mathcal{G}_K^{C,I}\}$. $\square$

4. Reducts in the classical case

In this section, we will recall the attribute reduction theory in formal concept analysis in the classical case, we will consider the adaptation of the set $\text{Atg}(C)$ (Definition 13) to this particular case and we will present different interesting properties.

Taking into account the irreducible elements of a concept lattice, Ganter and Wille proved several results about attribute and object reduction in [14]. The following result obtained from the ones given in [14] and introduced in [19] characterizes the $\wedge$-irreducible elements of $\mathcal{B}(A, B, R)$.

Proposition 33 ([19]). *Let (A, B, R) be a formal context. The set of $\wedge$-irreducible elements of the concept lattice $\mathcal{B}(A, B, R)$ is:*

$$M_F(A) = \left\{(a^{\downarrow}, a^{\downarrow\uparrow}) \mid a^{\downarrow} \neq \bigcap\{a_i^{\downarrow} \mid a^{\downarrow} \subset a_i^{\downarrow}\}\right\}$$

The absolutely necessary, relatively necessary and absolutely unnecessary attributes are characterized from the $\wedge$-irreducible elements of $\mathcal{B}(A, B, R)$ as the following theorem shows.

Theorem 34 ([19]). *Given a formal context (A, B, R), the following equivalences are obtained:*

1. *$a \in I_f$ if and only if $(a^{\downarrow}, a^{\downarrow\uparrow}) \notin M_F(A)$.*
2. *$a \in K_f$ if and only if $(a^{\downarrow}, a^{\downarrow\uparrow}) \in M_F(A)$ and there exists $a_1 \in A$, $a_1 \neq a$, such that $(a_1^{\downarrow}, a_1^{\downarrow\uparrow}) = (a^{\downarrow}, a^{\downarrow\uparrow})$.*
3. *$a \in C_f$ if and only if $(a^{\downarrow}, a^{\downarrow\uparrow}) \in M_F(A)$ and $(a_1^{\downarrow}, a_1^{\downarrow\uparrow}) \neq (a^{\downarrow}, a^{\downarrow\uparrow})$, for all $a_1 \in A$, $a_1 \neq a$.*

When we consider FCA in the classical framework, each attribute of our context generates only one concept (unlike it happens when we are working in a fuzzy framework), this fact gives rise to remarkable differences to obtain reducts.

Obviously, in a similar way to the fuzzy case, all the reducts contain all absolutely necessary attributes. The differences between two different reducts are given by the choice of the relatively necessary attributes, which are shown in the following.

Firstly, we will adapt Definition 13 to the classical case.

Definition 35. *Given a context* (A, B, R) *and a concept* C *of* $(\mathcal{M}, \preceq)$*, we define* the set of attributes generating C *as:*

$$Atg(C) = \{a \in A \mid (a^{\downarrow}, a^{\downarrow\uparrow}) = C\}$$

Taking into account this definition, Theorem 34 can be rewritten as follows.

Proposition 36. *Given a formal context* (A, B, R)*, the following equivalences are obtained:*

1. $a \in I_f$ *if and only if it does not exist a concept* $C \in M_F(A)$ *such that* $a \in Atg(C)$.
2. $a \in K_f$ *if and only if there exists* $C \in M_F(A)$ *such that* $a \in Atg(C)$ *and* $card(Atg(C)) > 1$.
3. $a \in C_f$ *if and only if there exists* $C \in M_F(A)$ *such that* $a \in Atg(C)$ *and* $card(Atg(C)) = 1$.

The following result is a direct consequence of the fact that each attribute of the context generates only one concept of the lattice.

Corollary 37. *If* C *is a meet-irreducible concept and* $Atg(C) \cap K_f \neq \varnothing$*, then* $Atg(C) \subseteq K_f$.

Observe that in the fuzzy case this result does not hold, in general. It can be seen in Example 15, where $\text{Atg}(C_1) \cap K_f = \{a_2, a_3\} \subseteq K_f$ but $\text{Atg}(C_2) \not\subseteq K_f$, since $a_1, a_6 \in \text{Atg}(C_2)$ satisfying that $a_1 \in I_f$ and $a_6 \in C_f$.

The next results also reveal other important differences found in the classical framework. The proofs of these propositions are direct from the previous results, hence, they will be omitted.

Proposition 38. *Given a meet-irreducible concept* C*, we have that* $Atg(C) \cap C_f = \varnothing$ *if and only if* $Atg(C) \subseteq K_f$*.*

Proposition 39. *The set* $G_K = \{Atg(C) \mid C \in M_F(A) \quad and \quad Atg(C) \subseteq K_f\}$ *is always a partition of* K_f*.*

As a consequence of the previous results, we can deduce that the cardinality of the reducts of a context (A, B, R) in the classical case is always the same, that is, all the reducts have composed by the same number of attributes.

Proposition 40. *Given* $G_K = \{Atg(C) \mid C \in M_F(A, B, R) \quad and \quad Atg(C) \subseteq K_f\}$ *and any reduct* Y *of the context* (A, B, R)*. Then, the following equality is always satisfied:*

$$card(Y) = card(C_f) + card(G_K)$$

Proof: Since G_K is a partition of K_f (Proposition 39), the equality straightforwardly follows from Theorem 25. □

This last result is the main difference that we find when we compare the construction of reducts in the classical and in the fuzzy case. This fact shows also the main difficulty in order to build the reducts in the fuzzy case, in particular, in the multi-adjoint concept lattice framework.

5. Conclusions and future work

This paper analyzes the process of constructing reducts in the multi-adjoint concept lattice framework, emphasizing the relevance of the selection of relatively necessary attributes for building reducts. In order to carry out this work, we have studied the attributes that generate $\wedge$-irreducible concepts and several properties of this kind of attributes, considering the attribute classification theorems introduced in [8].

In addition, an adaptation of the attribute classification theorems, based on the definition of the attribute generating meet-irreducible concepts, has been presented. This adaptation provides a simplification of the original theorems, since makes them easier to understand and apply.

A study about the cardinality of reducts has provided a sufficient condition to guarantee all reducts of a multi-adjoint context have the same cardinality. Moreover, we have also determined a lower and an upper bound of the cardinality for any reduct.

All the results included in this work have been clarified by means of illustrative examples. Finally, the introduced results have been considered in the classical case from which we have obtained, for example, that all the reducts have the same cardinality and what is this value. This study will be essential in order to compute the reducts in the multi-adjoint concept lattice framework and also in other (fuzzy) FCA frameworks. Moreover, we will study in the future how these results can be used in (fuzzy) RST.

Furthermore, we will study more properties in order to know how we should select the relatively necessary attributes, what is the most efficient way to perform this process or how we can get reducts with a minimal number of attributes. Moreover, we are interested in obtaining an efficient mechanism to compute all possible reducts for any multi-adjoint context, identifying the sets with minimal number of attributes.

References

[1] C. Alcalde, A. Burusco, H. Bustince, A. Jurio, and J. A. Sanz. Evolution in time of L-fuzzy context sequences. *Information Sciences*, 326:202 – 214, 2016.

[2] C. Alcalde, A. Burusco, and R. Fuentes-González. Application of the L-fuzzy concept analysis in the morphological image and signal processing. *Annals of Mathematics and Artificial Intelligence*, 72(1-2):115–128, 2014.

[3] L. Antoni, S. Krajči, and O. Krídlo. Constraint heterogeneous concept lattices and concept lattices with heterogeneous hedges. *Fuzzy Sets and Systems*, 303:21 – 37, 2016.

[4] R. Bělohlávek. Concept lattices and order in fuzzy logic. *Annals of Pure and Applied Logic*, 128:277–298, 2004.

[5] A. Burusco and R. Fuentes-González. Construction of the L-fuzzy concept lattice. *Fuzzy Sets and Systems*, 97(1):109–114, 1998.

[6] J. Chen, J. Li, Y. Lin, G. Lin, and Z. Ma. Relations of reduction between covering generalized rough sets and concept lattices. *Information Sciences*, 304:16–27, 2015.

[7] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. On the classification of fuzzy-attributes in multi-adjoint concept lattices. *Lecture Notes in Computer Science*, 7903:266–277, 2013.

[8] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Attribute reduction in multi-adjoint concept lattices. *Information Sciences*, 294:41 – 56, 2015.

[9] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Multi-adjoint algebras versus extended-order algebras. *Applied Mathematics & Information Sciences*, 9(2L):365–372, 2015.

[10] M. E. Cornejo, J. Medina, and E. Ramírez-Poussa. Multi-adjoint algebras versus non-commutative residuated structures. *International Journal of Approximate Reasoning*, 66:119–138, 2015.

[11] C. Cornelis, R. Jensen, G. Hurtado, and D. Ślęzak. Attribute selection with fuzzy decision reducts. *Information Sciences*, 180:209–224, 2010.

[12] W. S. Du and B. Q. Hu. Attribute reduction in ordered decision tables via evidence theory. *Information Sciences*, 364-365:91 – 110, 2016.

[13] S. Elloumi, J. Jaam, A. Hasnah, A. Jaoua, and I. Nafkha. A multi-level conceptual data reduction approach based on the lukasiewicz implication. *Information Sciences*, 163(4):253 – 262, 2004. Information Technology.

[14] B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundation.* Springer Verlag, 1999.

[15] G. Georgescu and A. Popescu. Non-commutative fuzzy Galois connections. *Soft Computing*, 7(7):458–467, 2003.

[16] P. Hájek. *Metamathematics of Fuzzy Logic.* Trends in Logic. Kluwer Academic, 1998.

[17] R. Jensen and C. Cornelis. Fuzzy-rough nearest neighbour classification and prediction. *Theoretical Computer Science*, 412(42):5871–5884, 2011.

[18] S. Krajči. A generalized concept lattice. *Logic Journal of IGPL*, 13(5):543–550, 2005.

[19] J. Medina. Relating attribute reduction in formal, object-oriented and property-oriented concept lattices. *Computers & Mathematics with Applications*, 64(6):1992–2002, 2012.

[20] J. Medina and M. Ojeda-Aciego. Multi-adjoint t-concept lattices. *Information Sciences*, 180(5):712–725, 2010.

[21] J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.

[22] Z. Pawlak. Rough sets. *International Journal of Computer and Information Science*, 11:341–356, 1982.

[23] R. Ren and L. Wei. The attribute reductions of three-way concept lattices. *Know.-Based Syst.*, 99(C):92–102, May 2016.

[24] M.-W. Shao and K.-W. Li. Attribute reduction in generalized one-sided formal contexts. *Information Sciences*, 378:317 – 327, 2017.

[25] M.-W. Shao, H.-Z. Yang, and W.-Z. Wu. Knowledge reduction in formal fuzzy contexts. *Knowledge-Based Systems*, 73:265 – 275, 2015.

[26] L. Wei and J.-J. Qi. Relation between concept lattice reduction and rough set reduction. *Knowledge-Based Systems*, 23(8):934–938, 2010.