Algebraic structure and characterization of adjoint triples[⋆]

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Abstract

Implications pairs, adjoint pairs and adjoint triples provide general residuated structures considered in different mathematical theories. In this paper, we carry out a deep study on the operators involved in these structures, showing how they are characterized by means of the irreducible elements of a complete lattice. Moreover, the structure of each class of these operators will be analyzed. As a consequence, the use of these operators in real problems will be more tractable, fostering their consideration as basic and useful operators for providing, for instance, preferences among attributes and objects in a given database.

Keywords: adjoint triples, residuated operators, complete lattices

1. Introduction

Adjoint triples arise as an interesting generalization of t-norms and their residuated implications, since they preserve their main properties and retain only the minimal mathematical requirements for guaranteeing operability. Specifically, they are tuples composed of an adjoint conjunctor and two residuated implications. The fact of requiring few restrictions broadens considerably the fields of application of adjoint triples, for example, they can be used in non-commutative and/or non-associative environments. Undoubtedly, knowing in

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depth these operators will allow us to solve a larger number of real problems, which clearly highlight the importance of studying this kind of operators. In addition, these operators were considered for the first time in logic programming, in order to present a fuzzy general framework in this theory. Due to different adjoint triples were used in the aforementioned framework, it was called multi-adjoint logic programming [31, 32]. Multi-adjoint logic programming was extended to a non-monotonic framework in [6, 7]. Following this philosophy, adjoint triples were introduced in two mathematical theories for analyzing databases, formal concept analysis and rough set theory, giving rise to multi-adjoint concept lattices and multi-adjoint fuzzy rough sets [14, 15, 30]. In addition, these operators were considered in fuzzy relation equations providing a new extension of these equations [8, 19, 20], which allows to solve more general problems. Recently, adjoint triples have been also considered in fuzzy mathematical morphology [1, 2, 28]. For example, they were considered in [28] in order to extend the definition of fuzzy relational erosions and dilations to handle membership values. The number of mathematical theories, in which adjoint triples has been applied, also justifies the study of these operators.

An interesting study on adjoint triples which shows important properties of these operators was carried out in [9]. An intense comparison among different general (non-commutative) algebraic structures such as sup-preserving aggregations [3], quantales [5, 25], u-norms [27], uninorms [22, 35] and general implications considered in extended-order algebras [17, 18, 24], was given in [12]. Specifically, it was proven that the aforementioned operators are particular cases of the adjoint conjunctors, when they have a residuated implication.

It is worth noting that it is not always necessary to consider all the operators that make up an adjoint triple [21]. For instance, only the residuated implications are needed in order to define the concept-forming operators within the theory of fuzzy formal concept analysis. Whereas in the generalization of rough set theory given by multi-adjoint object-oriented concept lattices and multi-adjoint property-oriented concept lattices [29], the adjoint conjunctor and only one residuated implication are used to define concept-forming operators. Therefore, it is also interesting the study of these operators by pairs, that is, implications pairs and adjoint pairs [12].

The goal of this paper is to know better implications pairs, adjoint pairs and adjoint triples, which will lead us consequently to know better other operators generalized by them. This paper characterizes implications pairs, adjoint pairs and adjoint triples, explaining how they are composed of and how they are related among them. Moreover, a hierarchy among them will be established and the obtained algebraic structure will be studied. This characterization together with the hierarchy provides some relevant advantages in real applications. Different examples are included in order to illustrate the developed theory.

The paper ir organized as follows. Section 2 includes some preliminary notions related to lattice theory. Section 3 provides the main notions and properties associated with implications pairs, adjoint pairs and adjoint triples. The characterization of these operators is introduced in Section 4. The algebraic structure formed by these operators is analyzed in Section 5. The paper finishes with some conclusions and prospects for future work.

2. Lattice theory

Lattice theory is an interesting branch of modern algebra, which has gained a lot of popularity since it provides a unifying framework in many mathematical disciplines [4]. A brief summary with several notions and properties related to lattice theory will be presented in order to make the paper self-contained.

From now on, we will consider a lattice $(L, \preceq)$ where $\wedge, \vee$ are the meet and the join operators. Given an arbitrary subset $X \subseteq L$, we will denote the supremum X as either $\sup(X)$ or $\bigvee X$ and the infimum of X as either $\inf(X)$ or $\bigwedge X$.

Definition 1 ([16]). Let $(L, \preceq)$ be a lattice and $M \subseteq L$ a non-empty subset. Then $(M, \preceq)$ is a *sublattice* of $(L, \preceq)$, if for each $a, b \in M$ we have that $a \vee b \in M$ and $a \wedge b \in M$.

Lemma 2 ([16]). *A complete join (meet) semilattice* $(L, \preceq)$ *with a minimum element (maximum element) is a complete lattice.*

Now, we present the notion of join-irreducible element which will play an important role in this paper.

Definition 3 ([16]). Given a lattice $(L, \preceq)$ and an element $x \in L$ verifying that:

1. If L has a bottom element $\perp$, then $x \neq \perp$.
2. If $x = y \vee z$, then $x = y$ or $x = z$, for all $y, z \in L$.

we call x *join-irreducible ($\vee$-irreducible) element* of L. Condition (2) is equivalent to

2′ If $y < x$ and $z < x$, then $y \vee z < x$, for all $y, z \in L$.

Hence, if x is $\vee$-irreducible, then it cannot be represented as the supremum of strictly smaller elements. A *meet-irreducible ($\wedge$-irreducible) element* of L is defined dually.

Note that, in a finite lattice, each element is equal to the supremum of join-irreducible elements [16]. The next definition shows when the decomposition of an element of a lattice as the supremum of join-irreducible elements is irredundant.

Definition 4 ([4]). Given a lattice $(L, \preceq)$ and an element $x \in L$, if there are join-irreducible elements $y_1, y_2, \ldots, y_n$, such that $x = y_1 \vee y_2 \vee \cdots \vee y_n$, then we say that x has a *finite* $\vee$*-decomposition*. Moreover, if for each $i \in \{1, \ldots, n\}$, $x \neq y_1 \vee \ldots y_{i-1} \vee y_{i+1} \vee \cdots \vee y_n$, then the decomposition is called *irredundant*, and we say that x has an *irredundant finite* $\vee$*-decomposition*.

In the following, we recall the notion of descending chain condition.

Definition 5 ([16]). Let $(P, \leq)$ be an ordered set. We say that P satisfies the *descending chain condition*, if given any sequence $\cdots \leq x_n \leq \cdots \leq x_2 \leq x_1$ of elements of P, there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \ldots$. The dual of the descending chain condition is the *ascending chain condition*.

Specifically, in a lattice satisfying the descending chain condition, each element of the lattice can be expressed from all join-irreducible elements lesser or equal than it.

Proposition 6 ([16]). *Let $(L, \preceq)$ be a lattice which satisfies the descending chain condition and $\mathcal{J}(L)$ be the set of join-irreducible elements of the lattice. Then, the following statement holds for all $a \in L$:*

$$a = \bigvee \{x \in \mathcal{J}(L) \mid x \preceq a\}$$

A dual result related to the meet-irreducible elements of a lattice satisfying the ascending chain condition can be easily obtained.

The following technical result of lattices satisfying the ascending chain condition will be useful at the end of this section.

Proposition 7 ([16]). *If $(L, \preceq)$ is a lattice satisfying the ascending chain condition, then for every non-empty subset A of L there exists a finite subset F of A such that $\bigvee A = \bigvee F$.*

Another definition which will play a fundamental role in the main results of this paper is the notion of distributive lattice.

Definition 8 ([16]). A lattice $(L, \preceq)$ is called *distributive* if the following equality is satisfied for all $x, y, z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Observe that the above condition is equivalent to its dual expression:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

A characterization of non-distributive lattices, from the lattices M_3 and N_5 (see Figure 1), was also given in [16].

Theorem 9 ([16]). *A lattice $(L, \preceq)$ is non-distributive if and only if M_3 or N_5 is a sublattice of $(L, \preceq)$.*

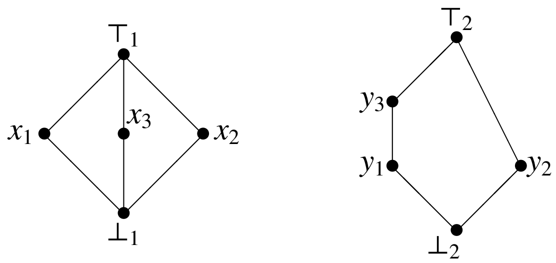

Figure 1: Examples of non distributive lattices: M_3 (left side) and N_5 (right side)

The next results show interesting properties related to the notion of irredundant finite $\vee$-decomposition in distributive lattices.

Lemma 10 ([4]). *In a distributive lattice, the representation of an element as an irredundant finite $\vee$-decomposition is unique.*

If the descending chain condition also holds, then the existence of such decomposition for each element of the lattice can be proven. Dually, if the ascending chain condition is satisfied.

Theorem 11 ([4]). *In a distributive lattice which satisfies the descending chain condition, each element has a unique irredundant finite $\vee$-decomposition.*

Finally, we present a property associated with the join-irreducible elements of a distributive lattice.

Lemma 12 ([4]). *In a distributive lattice* $(L, \preceq)$*, if* p *is a join-irreducible element and* $p \preceq \bigvee_{i=1}^{n} x_i$ *then there exists* $k \in \{1, \dots, n\}$ *such that* $p \preceq x_k$*.*

Notice that, applying Proposition 7, we obtain a similar property to the one given in Lemma 12 to the infinite case.

Lemma 13. *In a distributive lattice satisfying the ascending chain condition* $(L, \preceq)$*, if* p *is a join-irreducible element and* $p \preceq \bigvee_{i \in I} x_i$*, then there exists* $i \in I$ *such that* $p \preceq x_i$*.*

A dual property to the one given in the previous lemma is verified by the meet-irreducible elements of a distributive lattice.

3. Implications pairs, adjoint pairs and adjoint triples

Adjoint triples are a generalization of triangular norms and their residuated implications. These operators play a crucial role as basic calculus operators in different frameworks such as multi-adjoint logic programming, multi-adjoint concept lattices, multi-adjoint fuzzy rough sets and multi-adjoint fuzzy relation equations. This fact is due to adjoint triples increase the flexibility of these previous frameworks, for example, when conjunctors are required to be neither commutative nor associative, which is a worthy feature nowadays. The formal definition of adjoint triple is given in the following definition.

Definition 14. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\& \colon P_1 \times P_2 \to P_3$, $\swarrow \colon P_3 \times P_2 \to P_1$, $\nwarrow \colon P_3 \times P_1 \to P_2$ be mappings. We say that $(\&, \swarrow, \nwarrow)$ is an *adjoint triple* with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ if the following double equivalence is satisfied:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x \tag{1}$$

for all $x \in P_1$, $y \in P_2$ and $z \in P_3$. The previous double equivalence is called *adjoint property*.

The following properties corresponding to the operators $\&$, $\swarrow$ and $\nwarrow$ are obtained as a direct consequence of the adjoint property.

Proposition 15 ([12]). *Let* $(\&, \swarrow, \nwarrow)$ *be an adjoint triple with respect to the posets* $(P_1, \leq_1)$*,* $(P_2, \leq_2)$ *and* $(P_3, \leq_3)$*, then the following properties are satisfied:*

1. $\&$ *is order-preserving on both arguments.*
2. $\swarrow$ *and* $\nwarrow$ *are order-preserving on the first argument and order-reversing on the second argument.*
3. $\bot_1 \,\&\, y = \bot_3$, $\top_3 \swarrow y = \top_1$*, for all* $y \in P_2$*, when* $(P_1, \leq_1, \bot_1, \top_1)$ *and* $(P_3, \leq_3, \bot_3, \top_3)$ *are bounded posets.*
4. $x \,\&\, \bot_2 = \bot_3$ *and* $\top_3 \nwarrow x = \top_2$*, for all* $x \in P_1$*, when* $(P_2, \leq_2, \bot_2, \top_2)$ *and* $(P_3, \leq_3, \bot_3, \top_3)$ *are bounded posets.*
5. $z \nwarrow \bot_1 = \top_2$ *and* $z \swarrow \bot_2 = \top_1$*, for all* $z \in P_3$*, when* $(P_1, \leq_1, \bot_1, \top_1)$ *and* $(P_2, \leq_2, \bot_2, \top_2)$ *are bounded posets.*
6. *When the supremum and the infimum exist:*
   (a) $\left(\bigvee_{x' \in X} x'\right) \& \, y = \bigvee_{x' \in X} (x' \,\&\, y)$*, for all* $X \subseteq P_1$ *and* $y \in P_2$*.*
   (b) $\left(\bigwedge_{z' \in Z} z'\right) \swarrow y = \bigwedge_{z' \in Z} (z' \swarrow y)$*, for any* $Z \subseteq P_3$ *and* $y \in P_2$*.*
   (c) $x \,\&\, \left(\bigvee_{y' \in Y} y'\right) = \bigvee_{y' \in Y} (x \,\&\, y')$*, for all* $Y \subseteq P_2$ *and* $x \in P_1$*.*
   (d) $\left(\bigwedge_{z' \in Z} z'\right) \nwarrow x = \bigwedge_{z' \in Z} (z' \nwarrow x)$*, for all* $Z \subseteq P_3$ *and* $x \in P_1$*.*
   (e) $z \swarrow \left(\bigvee_{y' \in Y} y'\right) = \bigwedge_{y' \in Y} (z \swarrow y')$*, for all* $Y \subseteq P_2$ *and* $z \in P_3$*.*
   (f) $z \nwarrow \left(\bigvee_{x' \in X} x'\right) = \bigwedge_{x' \in X} (z \nwarrow x')$*, for all* $X \subseteq P_1$ *and* $z \in P_3$*.*
7. $z \swarrow y = \max\{x \in P_1 \mid x \,\&\, y \leq_3 z\}$*, for all* $y \in P_2$ *and* $z \in P_3$*.*
8. $z \nwarrow x = \max\{y \in P_2 \mid x \,\&\, y \leq_3 z\}$*, for all* $x \in P_1$ *and* $z \in P_3$*.*
9. $x \,\&\, y = \min\{z \in P_3 \mid x \leq_1 z \swarrow y\} = \min\{z \in P_3 \mid y \leq_2 z \nwarrow x\}$*, for all* $x \in P_1$ *and* $y \in P_2$*.*

Observe that the properties shown in Proposition 15 are always satisfied if $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$ are complete lattices. Indeed, in this case, by items 6(a) and 6(b) in Proposition 15, the Freyd's adjoint functor theorem [26, 33] shows that the mappings $\&^y \colon P_1 \to P_3$ and $\swarrow^y \colon P_3 \to P_1$, defined for each $x \in P_1$, $z \in P_3$ as $\&^y(x) = x \,\&\, y$ and $\swarrow^y (z) = z \swarrow y$, form an adjunction for all $y \in P_2$. Similarly, items 6(c) and 6(d) implies that ${}^x\& \colon P_2 \to P_3$ and $\nwarrow_x \colon P_3 \to P_2$, defined for each $y \in P_2$, $z \in P_3$ as ${}^x\&(y) = x \,\&\, y$ and $\nwarrow_x (z) = z \nwarrow y$, also form an adjunction for all $x \in P_1$ in the framework of complete lattices. The following result complements this comment.

Proposition 16. *Given the complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$, an arbitrary operator $\&\colon L_1 \times L_2 \to L_3$ and the mappings $\swarrow\colon L_3 \times L_2 \to L_1$, $\nwarrow\colon L_3 \times L_1 \to L_2$, defined as $z \swarrow y = \sup\{x' \in L_1 \mid x' \& y \preceq_3 z\}$ and $z \nwarrow x = \sup\{y' \in L_2 \mid x \& y' \preceq_3 z\}$, respectively, for all $x \in L_1$, $y \in L_2$ and $z \in L_3$, the next statements are equivalent:*

1. *$(\&, \swarrow, \nwarrow)$ is an adjoint triple with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$.*
2. *For all $x \in L_1$, $X \subseteq L_1$, $y \in L_2$ and $Y \subseteq L_2$,*

$$\left(\bigvee_{x' \in X} x'\right) \& y = \bigvee_{x' \in X} (x' \& y) \quad \textit{and} \quad x \& \left(\bigvee_{y' \in Y} y'\right) = \bigvee_{y' \in Y} (x \& y')$$

3. *$z \swarrow y = \max\{x' \in L_1 \mid x' \& y \preceq_3 z\}$ and $z \nwarrow x = \max\{y' \in L_2 \mid x \& y' \preceq_3 z\}$ for all $x \in L_1$, $y \in L_2$ and $z \in L_3$, and $\&$ is order-preserving on both arguments.*

Proof. The equivalence "(1) if and only if (2)" follows from Freyd's adjoint functor theorem, considering the mappings commented above. Hence, the proof will be based on item 3. First of all, "(2) implies (3)" will be proved and then the proof will finish with the proof of "(3) implies (1)". Consider the operators $\swarrow\colon L_3 \times L_2 \to L_1$, $\nwarrow\colon L_3 \times L_1 \to L_2$, defined as $z \swarrow y = \sup\{x' \in L_1 \mid x' \& y \preceq_3 z\}$ and $z \nwarrow x = \sup\{y' \in L_2 \mid x \& y' \preceq_3 z\}$, respectively, for all $x \in L_1$, $y \in L_2$ and $z \in L_3$. In addition, for each $y \in L_2$ and $z \in L_3$, we define the sets $X = \{x' \in L_1 \mid x' \& y \preceq_3 z\}$ and $Y = \{y' \in L_2 \mid x \& y' \preceq_3 z\}$.

(2) implies (3): Taking into account Statement (2) and the definitions of the operators $\swarrow$, $\nwarrow$, we can ensure that the following chains of inequalities hold, for all $x \in L_1$, $y \in L_2$ and $z \in L_3$:

$$(z \swarrow y) \& y = \left(\bigvee_{x' \in X} x'\right) \& y = \bigvee_{x' \in X} (x' \& y) \preceq_3 z$$

$$x \& (z \nwarrow x) = x \& \left(\bigvee_{y' \in Y} y'\right) = \bigvee_{y' \in Y} (x \& y') \preceq_3 z$$

Therefore, $z \swarrow y \in X$ and $z \nwarrow x \in Y$. As a consequence, we deduce that the equalities $z \swarrow y = \max\{x' \in L_1 \mid x' \& y \preceq_3 z\}$ and $z \nwarrow x = \max\{y' \in L_2 \mid x \& y' \preceq_3 z\}$ are satisfied, for all $x \in L_1$, $y \in L_2$ and $z \in L_3$.

Now, we will prove that $\&$ is an order-preserving operator on the first argument. Given $x_1, x_2 \in L_1$, suppose that $x_1 \preceq_1 x_2$. Applying Statement

(2) and the supremum property, we can deduce that $x_1 \mathbin{\&} y \preceq_3 (x_1 \mathbin{\&} y) \vee (x_2 \mathbin{\&} y) = (x_1 \vee x_2) \mathbin{\&} y = x_2 \mathbin{\&} y$, for all $y \in L_2$. Following a similar reasoning, we obtain the inequality $x \mathbin{\&} y_1 \preceq_3 x \mathbin{\&} y_2$, where $x \in L_1$ and $y_1, y_2 \in L_2$, verifying that $y_1 \preceq_2 y_2$. Hence, $\&$ is order-preserving on both arguments.

(3) implies (1): We will prove the double equivalence $x \preceq_1 z \swarrow y$ iff $x \mathbin{\&} y \preceq_3 z$ iff $y \preceq_2 z \nwarrow x$ holds, where $x \in L_1$, $y \in L_2$ and $z \in L_3$.

Suppose that $x \preceq_1 z \swarrow y$, as $\&$ is order-preserving on the first argument, we obtain that $x \mathbin{\&} y \preceq_3 (z \swarrow y) \mathbin{\&} y$ holds. In addition, we have that $z \swarrow y \in X$ because $z \swarrow y = \max\{x' \in L_1 \mid x' \mathbin{\&} y \preceq_3 z\}$. Hence, $(z \swarrow y) \mathbin{\&} y \preceq_3 z$ and consequently $x \mathbin{\&} y \preceq_3 z$.

Conversely, assume that $x \mathbin{\&} y \preceq_3 z$. Clearly, $x \in X$. By Statement (3), $z \swarrow y = \max\{x' \in L_1 \mid x' \mathbin{\&} y \preceq_3 z\}$ and therefore, $x \preceq_1 z \swarrow y$.

The another equivalence follows similarly. Therefore, we conclude that $(\&, \swarrow, \nwarrow)$ is an adjoint triple with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$. $\square$

The most usual adjoint triples with respect to $([0,1], \leq)$ are those defined from the Gödel, product and Łukasiewicz t-norms together with their residuated implications. Due to these t-norms are commutative, we can ensure that $\swarrow^G = \nwarrow_G$, $\swarrow^P = \nwarrow_P$ and $\swarrow^Ł = \nwarrow_Ł$. The mentioned adjoint triples are given below:

$$\&_G(x, y) = \min\{x, y\} \qquad z \swarrow^G y = \begin{cases} 1 & \text{if } y \leq z \\ z & \text{otherwise} \end{cases}$$

$$\&_P(x, y) = x \cdot y \qquad z \swarrow^P y = \min\{1, z/y\}$$

$$\&_Ł(x, y) = \max\{0, x + y - 1\} \qquad z \swarrow^Ł y = \min\{1, 1 - y + z\}$$

Other general examples of adjoint triples were given for instance in [9]. There exist different cases in which considering only pairs of operators satisfying an adjoint property is sufficient and provides more flexibility [21]. For instance, the multi-adjoint concept lattice framework is a particular environment in which it is not necessary to assume adjoint triples but pairs. Specifically, the basic operators needed in the definition of the concept-forming operators [10, 30] are Galois implications pairs, as was justified in [21]. These operators and other possible pairs are introduced below.

Definition 17. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets and $\& \colon P_1 \times P_2 \to P_3$, $\swarrow \colon P_3 \times P_2 \to P_1$, $\nwarrow \colon P_3 \times P_1 \to P_2$ be mappings. We say that:

- $(\&, \swarrow)$ is a *right adjoint pair* with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ if the next equivalence is satisfied, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z$$

- $(\&, \nwarrow)$ is a *left adjoint pair* with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ if the next equivalence is verified, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$:

$$x \,\&\, y \leq_3 z \quad \text{iff} \quad y \leq_2 z \nwarrow x$$

- $(\swarrow, \nwarrow)$ is a *Galois implications pair* with respect to $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ if the next equivalence is satisfied, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$:

$$x \leq_1 z \swarrow y \quad \text{iff} \quad y \leq_2 z \nwarrow x \tag{2}$$

Interesting properties related to left/right adjoint pairs and Galois implications pairs are deduced from these previous equivalences [12]. In the following, we will show properties related to Galois implications pairs obtained from Equivalence (2). Note that, similar properties can be obtained for left/right adjoint pairs in an analogous way.

Proposition 18 ([12]). *Let $(\swarrow, \nwarrow)$ be a Galois implications pair with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P_3, \leq_3)$, then the next properties are verified:*

1. $\swarrow$ *and* $\nwarrow$ *are order-reversing on the second argument.*
2. $z \nwarrow \bot_1 = \top_2$ *and* $z \swarrow \bot_2 = \top_1$ *for all $z \in P_3$, when $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_2, \leq_2, \bot_2, \top_2)$ are bounded posets.*
3. $x \leq_1 z \swarrow (z \nwarrow x)$ *and* $y \leq_2 z \nwarrow (z \swarrow y)$*, for all $x \in P_1$, $y \in P_2$, $z \in P_3$.*
4. $z \swarrow y = \max\{x \in P_1 \mid y \leq_2 z \nwarrow x\}$*, for all $y \in P_2$ and $z \in P_3$.*
5. $z \nwarrow x = \max\{y \in P_2 \mid x \leq_1 z \swarrow y\}$*, for all $x \in P_1$ and $z \in P_3$.*
6. *When the supremum and the infimum exist:*
   (a) $z \swarrow \left(\bigvee_{y' \in Y} y'\right) = \bigwedge_{y' \in Y} (z \swarrow y')$*, for all $Y \subseteq P_2$ and $z \in P_3$.*
   (b) $z \nwarrow \left(\bigvee_{x' \in X} x'\right) = \bigwedge_{x' \in X} (z \nwarrow x')$*, for all $X \subseteq P_1$ and $z \in P_3$.*

The next result presents the equivalences corresponding to Galois implications pairs when $(P_1, \leq_1)$ and $(P_2, \leq_2)$ are complete lattices, which is related to Proposition 16.

Proposition 19. *Let* $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$ *be two complete lattices,* $(P_3, \preceq_3)$ *be a poset,* $\nwarrow\colon P_3 \times L_1 \to L_2$ *be an arbitrary operator and* $\swarrow\colon P_3 \times L_2 \to L_1$*, defined as* $z \swarrow y = \sup\{x' \in L_1 \mid y \preceq_2 z \nwarrow x'\}$*, for all* $y \in L_2$ *and* $z \in P_3$*. The following statements are equivalent:*

1. $(\swarrow, \nwarrow)$ *is a Galois implications pair with respect to* $(L_1, \preceq_1)$*,* $(L_2, \preceq_2)$*,* $(P_3, \preceq_3)$*.*
2. $z \nwarrow \left(\bigvee_{x' \in X} x' \right) = \bigwedge_{x' \in X} (z \nwarrow x')$*, for all* $X \subseteq L_1$ *and* $z \in P_3$*.*
3. $z \swarrow y = \max\{x' \in L_1 \mid y \preceq_2 z \nwarrow x'\}$*, for all* $y \in L_2$ *and* $z \in P_3$*, and* $\nwarrow$ *is order-reversing on the second argument.*

Proof. The proof similarly follows as in Proposition 16.

The pair of operators shown in the following example forms a Galois implications pair. However, these operators are not the adjoint implications of an adjoint triple, as it was proven in [21].

Example 20. The pair $(\swarrow, \nwarrow)$ whose operators are defined on the complete lattice $([0,1], \leq)$ as follows:

$$z \swarrow y = \begin{cases} \dfrac{1-y}{1-z} & \text{if } z \leq \dfrac{1}{2} \text{ and } z < y \\ \sqrt{\dfrac{1-y}{1-z}} & \text{if } z > \dfrac{1}{2} \text{ and } z < y \\ 1 & \text{if } z \geq y \end{cases} \qquad z \nwarrow x = \begin{cases} 1 - x^2 \cdot (1-z) & \text{if } z < \dfrac{1}{2} \\ 1 - x \cdot (1-z) & \text{if } z \geq \dfrac{1}{2} \end{cases}$$

for all $x, y, z \in [0,1]$, form a Galois implications pair. The authors of [21] proved that there does not exist any conjunctor & such that $(\&, \swarrow, \nwarrow)$ is an adjoint triple. □

The previous properties associated with adjoint triples and Galois implications pairs will facilitate the development of the proofs and the examples carried out throughout this paper. A more detailed study of adjoint triples, left/right adjoint pairs and Galois implications pairs can be found in [9, 11, 12].

In the next section, we will investigate how we can define the conjunctor and the implication operators of adjoint triples, left/right adjoint pairs and Galois implications pairs, by using join-irreducible elements.

4. Defining pairs and triples by using join-irreducible elements

Irreducible elements are one of the most important elements of a lattice, for example, if a lattice satisfies the descending chain condition then the irreducible elements form a base from which we can obtain the complete lattice. We are interested in using this powerful property in order to provide a mechanism to define implication and conjunctor operators, which can form Galois implications pairs, left/right adjoint pairs and adjoint triples.

First of all, we will present the mechanism which allows us to define an implication operator of a Galois implications pair. To achieve this goal, will be fundamental to consider the infimum operator and require that each element of the lattice have a unique irredundant finite $\vee$-decomposition. This last condition will be guaranteed by the descending chain condition and the distributivity property.

Theorem 21. *Let $(L_1, \preceq_1)$ be a distributive complete lattice satisfying the descending chain condition, $(L_2, \preceq_2)$ a complete lattice, $(P_3, \leq_3)$ a poset and $\mathcal{J}(L_1)$ the set of join-irreducible elements of L_1. For each $z \in P_3$, we define the operator ${}^{z}\nwarrow\colon L_1 \to L_2$, for all $x \in L_1$, as follows:*

$$
{}^{z}\nwarrow(x) = \begin{cases} \top_2 & \textit{if} \quad x = \bot_1 \\ \displaystyle\bigwedge_{p \in J(x)} f_z(p) & \textit{otherwise} \end{cases} \tag{3}
$$

where $J(x)$ denotes the set of join-irreducible elements of the irredundant finite $\vee$-decomposition of each element x in L_1, and $f_z\colon \mathcal{J}(L_1) \to L_2$ is an order-reversing mapping[1]*. Then, the operator $\nwarrow\colon P_3 \times L_1 \to L_2$ defined as $z \nwarrow x = {}^{z}\nwarrow(x)$, for all $z \in P_3$ and $x \in L_1$, is an implication of a Galois implications pair.*

Proof. First of all, we have that the mapping $\nwarrow\colon P_3 \times L_1 \to L_2$ defined as $z \nwarrow x = {}^{z}\nwarrow(x)$, for all $z \in P_3$ and $x \in L_1$, is well-defined since, by Theorem 11, each element $x \in L_1$ has a unique irredundant finite $\vee$-decomposition.

In order to demonstrate that $\nwarrow$ is an implication of a Galois implications pair, we will prove that this implication satisfies the property given in Proposition 19(3). We will see that the operator $\swarrow\colon P_3 \times L_2 \to L_1$ defined as $z \swarrow y =$

[1]Note that, the irredundant finite $\vee$-decomposition of an join-irreducible element $p \in \mathcal{J}(L_1)$ is the proper element p, and therefore, ${}^{z}\nwarrow(p) = f_z(p)$ for all $p \in \mathcal{J}(L_1)$.

$\sup\{x \in L_1 \mid y \preceq_2 z \nwarrow x\}$ is actually a maximum, for all $y \in L_2$ and $z \in P_3$, and that $\nwarrow$ is order-reversing on the second argument.

On the one hand, we will see that the operator $\nwarrow$ is order-reversing on the second argument. That is, we will prove that if $x_1, x_2 \in L_1$ satisfying that $x_1 \preceq_1 x_2$, then $z \nwarrow x_2 \preceq_2 z \nwarrow x_1$, for all $z \in P_3$. Applying Theorem 11, we can express $x_1, x_2 \in L_1$ from their corresponding irredundant finite $\vee$-decompositions, that is, $x_1 = \bigvee_{p \in J(x_1)} p$ and $x_2 = \bigvee_{q \in J(x_2)} q$ where $J(x_1)$ and $J(x_2)$ denote the sets of join-irreducible elements of the unique irredundant finite $\vee$-decomposition of the elements x_1 and x_2, respectively.

Suppose that $x_1 \neq \bot_1$ and $x_2 \neq \bot_1$. Taking into account that $x_1 \preceq_1 x_2$, we have that $\bigvee_{p \in J(x_1)} p \preceq_1 \bigvee_{q \in J(x_2)} q$. Clearly, the inequality $p \preceq_1 x_1 = \bigvee_{p \in J(x_1)} p$ holds, for each $p \in J(x_1)$. Therefore, we can ensure that $p \preceq_1 \bigvee_{q \in J(x_2)} q$, for all $p \in J(x_1)$. Hence, by Lemma 12, for every $p \in J(x_1)$, there exists $q_p \in J(x_2)$, such that $p \preceq_1 q_p$. Since f_z is an order-reversing mapping, we obtain that for every $p \in J(x_1)$ there exists $q_p \in J(x_2)$ such that $f_z(q_p) \preceq_2 f_z(p)$. Consequently, by using the infimum property and the definition of the operators $\nwarrow$ and ${}^{z}\nwarrow$, we have the following chain of inequalities:

$$z \nwarrow x_2 = {}^{z}\nwarrow (x_2) = \bigwedge_{q \in J(x_2)} f_z(q) \preceq_2 \bigwedge_{p \in J(x_1)} f_z(q_p) \preceq_2 \bigwedge_{p \in J(x_1)} f_z(p) = {}^{z}\nwarrow (x_1) = z \nwarrow x_1$$

Now, without loss of generality, assume that $x_1 = \bot_1$ and $x_2 \neq \bot_1$. By Equation (3), the following equalities are obtained:

$$\begin{aligned} {}^{z}\nwarrow (x_1) &= {}^{z}\nwarrow (\bot_1) = \top_2 \\ {}^{z}\nwarrow (x_2) &= {}^{z}\nwarrow \left(\bigvee_{q \in J(x_2)} q \right) = \bigwedge_{q \in J(x_2)} f_z(q) \end{aligned}$$

Therefore, the inequality $z \nwarrow x_2 = {}^{z}\nwarrow (x_2) \preceq_2 {}^{z}\nwarrow (x_1) = z \nwarrow x_1$ is verified, for all $z \in P_3$. Notice that the inequality $z \nwarrow x_2 \preceq_2 z \nwarrow x_1$ is trivially obtained when $x_1 = x_2 = \bot_1$. Hence, we can conclude that $\nwarrow$ is order-reversing on the second argument.

Finally, it remains to prove that $z \swarrow y = \max\{x \in L_1 \mid y \preceq_2 z \nwarrow x\}$ holds, for all $y \in L_2$ and $z \in P_3$. We will prove, for each $y \in L_2$, that $\sup(X)$ where $X = \{p \in \mathcal{J}(L_1) \mid y \preceq_2 z \nwarrow p\}$ is this maximum. First of all, we will prove that $\sup(X) \in \{x \in L_1 \mid y \preceq_2 z \nwarrow x\}$. For that, we will distinguish two cases:

- If $X = \varnothing$, then $\sup(X) = \bot_1$. Clearly, by Equation (3), we have that the inequality $y \preceq_2 z \nwarrow \bot_1 = \top_2$ holds, for all $y \in L_2$ and $z \in P_3$. Consequently, we obtain that $\bot_1 \in \{x \in L_1 \mid y \preceq_2 z \nwarrow x\}$, for all $y \in L_2$ and $z \in P_3$.

- If $X \neq \varnothing$, then $\sup(X) = \sup\{p \in \mathcal{J}(L_1) \mid y \preceq_2 z \nwarrow p\} \neq \bot_1$ and X provides a finite $\vee$-decomposition of $\sup(X)$, that is, $\sup(X) = \bigvee_{p \in X} p$. Considering the irredundant finite $\vee$-decomposition of $\sup(X)$, that is, $\sup(X) = \bigvee_{p \in \Lambda} p$ with $\Lambda \subseteq X$, and applying Equation (3), we have that $z \nwarrow \sup(X) = {}^{z}\!\nwarrow (\sup(X)) = \bigwedge_{p \in \Lambda} f_z(p) = \bigwedge_{p \in \Lambda} {}^{z}\!\nwarrow (p) = \bigwedge_{p \in \Lambda} z \nwarrow p$ and therefore, since $y \preceq_2 z \nwarrow p$ holds for all $p \in \Lambda$, we can ensure that $y \preceq_2 z \nwarrow \sup(X)$ is satisfied for all $y \in L_2$ and $z \in P_3$. As a consequence, we obtain that $\sup(X) \in \{x \in L_1 \mid y \preceq_2 z \nwarrow x\}$, for all $y \in L_2$ and $z \in P_3$.

Now, we prove that $\sup(X)$ is the greatest element in the set $\{x \in L_1 \mid y \preceq_2 z \nwarrow x\}$. Specifically, we will prove that the inequality $x' \preceq_1 \sup(X)$ is verified, for all $x' \in \{x \in L_1 \mid y \preceq_2 z \nwarrow x\}$. Once again, we will distinguish two cases:

- If $x' = \bot_1$, then the inequality $\bot_1 \preceq_1 \sup(X)$ is trivially satisfied.
- If $x' \neq \bot_1$, we can express x' from its corresponding irredundant finite $\vee$-decomposition, that is, $x' = \bigvee_{p \in J(x')} p$ where $J(x')$ denotes the set of join-irreducible elements of the unique finite irredundant $\vee$-decomposition of the element x'. Taking into account that $x' \in \{x \in L_1 \mid y \preceq_2 z \nwarrow x\}$, by using Equation (3), we can deduce that:

$$y \preceq_2 z \nwarrow x' = {}^{z}\!\nwarrow (x') = {}^{z}\!\nwarrow \left(\bigvee_{p \in J(x')} p \right) = \bigwedge_{p \in J(x')} f_z(p) = \bigwedge_{p \in J(x')} {}^{z}\!\nwarrow (p) = \bigwedge_{p \in J(x')} z \nwarrow p$$

  because $z \nwarrow p = {}^{z}\!\nwarrow (p) = f_z(p)$, for all $p \in \mathcal{J}(L_1)$ and $z \in P_3$. By the infimum property, the inequality $y \preceq_2 z \nwarrow p$ is verifed, for all $p \in J(x')$. Hence, we obtain that $p \in X$, for all $p \in J(x')$. Finally, by the supremum property, we have that:

$$x' = \bigvee_{p \in J(x')} p \preceq_1 \bigvee_{x \in X} x = \sup(X)$$

Therefore, we can conclude that $z \swarrow y = \max\{x \in L_1 \mid y \preceq_2 z \nwarrow x\}$ holds, for all $y \in L_2$ and $z \in P_3$. □

The following example illustrates the previous results and the requirements given by the hypothesis. For example, it will show that the distributivity property is mandatory.

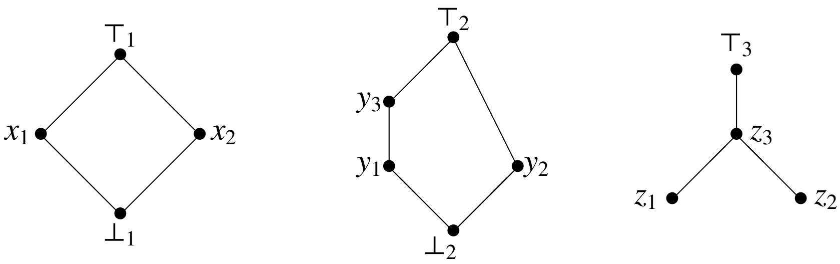

Figure 2: Lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and poset $(P_3, \preceq_3)$ of Example 22.

Example 22. Consider the distributive complete lattice $(L_1, \preceq_1)$, the complete lattice $(L_2, \preceq_2)$ and the poset $(P_3, \preceq_3)$ which are depicted in Figure 2 from left to right, respectively. Notice that, $(L_1, \preceq)$ satisfies the descending chain condition since it is a finite lattice. Taking into account the Hasse diagram, we can ensure that the set of join-irreducible elements of L_1 is $\mathcal{J}(L_1) = \{x_1, x_2\}$.

Now, we will compute some particular Galois implications pairs with respect to $(L_1, \preceq_1), (L_2, \preceq_2), (P_3, \preceq_3)$. For instance, we can consider the order-reversing mapping $f_z \colon \mathcal{J}(L_1) \to L_2$ defined as $f_z(x_1) = \bot_2$ and $f_z(x_2) = y_1$, for all $z \in P_3$. By using Equation (3) and Proposition 19(3), we can define the implication operators $\nwarrow$ and $\swarrow$ which are given by Table 1. In order to carry out the computations corresponding to the pair $(\swarrow, \nwarrow)$, we have considered the irredundant finite $\vee$-decompositions of the elements x_1, x_2 and $\top_1$. Since x_1 and x_2 are only obtained as supremum of themselves, we have that $z \nwarrow x_1 = {}^{z}\!\nwarrow(x_1) = f_z(x_1)$ and $z \nwarrow x_2 = {}^{z}\!\nwarrow(x_2) = f_z(x_2)$. The irredundant finite $\vee$-decomposition of the top element in L_1 is $\top_1 = \sup\{x_1, x_2\}$ and consequently, $z \nwarrow \top_1 = {}^{z}\!\nwarrow(\top_1) = \inf\{z \nwarrow x_1, z \nwarrow x_2\} = \inf\{{}^{z}\!\nwarrow(x_1), {}^{z}\!\nwarrow(x_2)\} = \inf\{f_z(x_1), f_z(x_2)\} = \bot_2$. Finally, applying Theorem 21, we obtain that the pair $(\swarrow, \nwarrow)$ displayed in Table 1 forms a Galois implications pair.

$\swarrow$	$\bot_2$	y_1	y_2	y_3	$\top_2$
z_1	$\top_1$	x_2	$\bot_1$	$\bot_1$	$\bot_1$
z_2	$\top_1$	x_2	$\bot_1$	$\bot_1$	$\bot_1$
z_3	$\top_1$	x_2	$\bot_1$	$\bot_1$	$\bot_1$
$\top_3$	$\top_1$	x_2	$\bot_1$	$\bot_1$	$\bot_1$

$\nwarrow$	$\bot_1$	x_1	x_2	$\top_1$
z_1	$\top_2$	$\bot_2$	y_1	$\bot_2$
z_2	$\top_2$	$\bot_2$	y_1	$\bot_2$
z_3	$\top_2$	$\bot_2$	y_1	$\bot_2$
$\top_3$	$\top_2$	$\bot_2$	y_1	$\bot_2$

Table 1: Definition of Galois implications pair $(\swarrow, \nwarrow)$ of Example 22.

Note that, it is not necessary to consider the same order-reversing mapping for each $z \in P_3$. For example, we can consider the previous order-reversing mapping for $z_1 \in P_3$ and other different mappings for the rest of values in P_3:

- $f_{z_1} : \mathcal{J}(L_1) \to L_2$ defined as $f_{z_1}(x_1) = \bot_2$ and $f_{z_1}(x_2) = y_1$.
- $f_{z_2} : \mathcal{J}(L_1) \to L_2$ defined as $f_{z_2}(x_1) = \bot_2$ and $f_{z_2}(x_2) = \top_2$.
- $f_{z_3} : \mathcal{J}(L_1) \to L_2$ defined as $f_{z_3}(x_1) = y_1$ and $f_{z_3}(x_2) = y_2$.
- $f_{\top_3} : \mathcal{J}(L_1) \to L_2$ defined as $f_{\top_3}(x_1) = y_1$ and $f_{\top_3}(x_2) = y_3$.

From these mappings and following the same procedure to the previously mentioned, Theorem 21 and Proposition 19(3) allow us to define the Galois implications pair $(\Swarrow, \Nwarrow)$ shown in Table 2.

$\Swarrow$	$\bot_2$	y_1	y_2	y_3	$\top_2$
z_1	$\top_1$	x_2	$\bot_1$	$\bot_1$	$\bot_1$
z_2	$\top_1$	x_2	x_2	x_2	x_2
z_3	$\top_1$	x_1	x_2	$\bot_1$	$\bot_1$
$\top_3$	$\top_1$	$\top_1$	$\bot_1$	x_2	$\bot_1$

$\Nwarrow$	$\bot_1$	x_1	x_2	$\top_1$
z_1	$\top_2$	$\bot_2$	y_1	$\bot_2$
z_2	$\top_2$	$\bot_2$	$\top_2$	$\bot_2$
z_3	$\top_2$	y_1	y_2	$\bot_2$
$\top_3$	$\top_2$	y_1	y_3	y_1

Table 2: Definition of Galois implications pair $(\Swarrow, \Nwarrow)$ of Example 22.

It is important to mention that the complete lattice L_2 is not required to be distributive. Indeed, $L_2 = N_5$ in this example. However, this requirement is mandatory with respect to the lattice L_1, since the mapping $\nwarrow$ is defined from the join-irreducible elements of L_1. For example, suppose that L_1 is the non-distributive lattice M_3 displayed in Figure 1. In this case, the set of join-irreducible elements of M_3 is given by $\mathcal{J}(M_3) = \{x_1, x_2, x_3\}$. As M_3 is a non-distributive lattice, we obtain different irredundant finite $\vee$-decompositions of the element $\top_1$. That is, $\top_1 = \sup\{x_1, x_2\}$, $\top_1 = \sup\{x_1, x_3\}$ and $\top_1 = \sup\{x_2, x_3\}$.

Now, we will consider the order-reversing mapping $f_z : \mathcal{J}(L_1) \to L_2$ defined as $f_z(x_1) = y_1$, $f_z(x_2) = \bot_2$ and $f_z(x_3) = y_3$, for all $z \in P_3$. Assuming the different irredundant finite $\vee$-decompositions of $\top_1$, we will conclude that the operator ${}^{z}\!\nwarrow$ obtained from Equation (3) is not well-defined. As a consequence, $\nwarrow$ is not well-defined.

On the one hand, if we consider the irredundant $\vee$-decomposition $\top_1 = \sup\{x_1, x_2\}$, we obtain ${}^{z}\!\nwarrow(\top_1) = \inf\{f_z(x_1), f_z(x_2)\} = \inf\{y_1, \bot_2\} = \bot_2$. On the other hand, if we consider the irredundant $\vee$-decomposition $\top_1 = \sup\{x_1, x_3\}$, we have that ${}^{z}\!\nwarrow(\top_1) = \inf\{f_z(x_1), f_z(x_3)\} = \inf\{y_1, y_3\} = y_1$.

Therefore, we can conclude that the mapping ${}^{z}\nwarrow$ is not well-defined and consequently $\nwarrow$ is not either. In order to avoid this problem, it is necessary to require the distributivity property to the lattice $(L_1, \preceq_1)$ in Theorem 21. $\square$

The following result shows that the counterpart of Theorem 21 holds.

Proposition 23. *Let $(L_1, \preceq_1)$ be a distributive complete lattice satisfying the descending chain condition, $(L_2, \preceq_2)$ a complete lattice and $(P_3, \preceq_3)$ a poset. For any Galois implications pair $(\swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(P_3, \preceq_3)$, the implication $\nwarrow$ satisfies Equation (3), for all $z \in P_3$.*

Proof. Given an arbitrary Galois implications pair $(\swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$, using Property (2) given in Proposition 18, we obtain that $z \nwarrow x = \top_2$ when $x = \bot_1$ and $z \in P_3$. From now on, we will assume that $x \neq \bot_1$. We can express x as an irredundant finite $\vee$-decomposition, since $(L_1, \preceq_1)$ is a distributive complete lattice satisfying the descending chain condition. Then, given $\mathcal{J}(L_1)$ the set of join-irreducible elements of L_1, we can write $x = \bigvee_{p \in J(x)} p$ where $J(x)$ denotes the set of join-irreducible elements of the irredundant finite $\vee$-decomposition of the element x. Notice that, by Theorem 11, such decomposition is unique. By the property (6b) given in Proposition 18, we obtain the following chain of equalities:

$$z \nwarrow x = z \nwarrow \left(\bigvee_{p \in J(x)} p \right) = \bigwedge_{p \in J(x)} \{z \nwarrow p\} = \bigwedge_{p \in J(x)} f_z(p)$$

where f_z is an order-reversing mapping defined as $f_z(p) = z \nwarrow p$, for all $p \in \mathcal{J}(L_1)$ and $z \in P_3$. Therefore, we can ensure that the implication $\nwarrow$ satisfies Equation (3). $\square$

A similar result arises for the implication $\swarrow$, if $(L_2, \preceq_2)$ is a distributive complete lattice satisfying the descending chain condition.

It is important to mention that, from Theorem 21 and Proposition 23, we can ensure that the number of Galois implications pairs which can be defined with respect to two finite distributive complete lattices satisfying the descending chain condition $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ and a finite poset $(P_3, \preceq_3)$, is $\operatorname{card}(P_3) \times \operatorname{card}(\mathcal{F})$, where $\mathcal{F} = \{f : \mathcal{J}(L_1) \to L_2 \mid f \text{ is an order-reversing mapping}\}$.

Following an analogous strategy to the one given in Theorem 21, the conjunctor operator of a right adjoint pair can be defined. We will also need that each element of the lattice have a unique irredundant finite $\vee$-decomposition. In this case, we will use the supremum operator instead of the infimum one.

Theorem 24. *Let $(L_1, \preceq_1)$ be a distributive complete lattice satisfying the descending chain condition, $(P_2, \preceq_2)$ a poset, $(L_3, \preceq_3)$ a complete lattice and $\mathcal{J}(L_1)$ the set of join-irreducible elements of L_1. For each $y \in P_2$, we define the operator $\&_y \colon L_1 \to L_3$, for all $x \in L_1$, as follows:*

$$\&_y(x) = \begin{cases} \perp_3 & \textit{if} \quad x = \perp_1 \\ \bigvee_{p \in J(x)} g_y(p) & \textit{otherwise} \end{cases} \tag{4}$$

where $J(x)$ denotes the set of join-irreducible elements of the irredundant finite $\vee$-decomposition of the elements x in L_1 and $g_y \colon \mathcal{J}(L_1) \to L_3$ is an order-preserving mapping[2]*. Then, the operator $\& \colon L_1 \times P_2 \to L_3$ defined as $x \,\&\, y = \&_y(x)$, for all $x \in L_1$ and $y \in P_2$, is the conjunctor of a right adjoint pair.*

Proof. The proof follows similarly as in Theorem 21. □

The counterpart of Theorem 24 is also satisfied as it is shown below.

Proposition 25. *Let $(L_1, \preceq_1)$ be a distributive complete lattice satisfying the descending chain condition, $(P_2, \preceq_2)$ a poset and $(L_3, \preceq_3)$ a distributive complete lattice. For any right adjoint pair $(\&, \swarrow)$ with respect to $(L_1, \preceq_1)$, $(P_2, \preceq_2)$, $(L_3, \preceq_3)$, the conjunctor $\&$ satisfies Equation* (4)*, for all $y \in P_2$.*

Proof. The proof can be deduced following a completely analogous reasoning to the one given in the proof of Proposition 23, by using Proposition 20 in [12], which is analogous to Proposition 18 for right adjoint pair.

It is also convenient to mention that, from Theorem 24 and Proposition 25, it is possible to guarantee the number of right adjoint pairs, which can be defined with respect to a finite distributive complete lattice satisfying the descending chain condition $(L_1, \preceq_1)$, a finite poset $(P_2, \preceq_2)$ and a finite distributive complete lattice $(L_3, \preceq_3)$, is $\mathrm{card}(P_2) \times \mathrm{card}(\mathcal{G})$, where $\mathcal{G} = \{g \colon \mathcal{J}(L_1) \to L_3 \mid g \text{ is an order-preserving mapping}\}$.

Notice that, given a poset $(P_1, \preceq_1)$, a distributive complete lattice satisfying the descending chain condition $(L_2, \preceq_2)$ and a complete lattice $(L_3, \preceq_3)$, we can also define the conjunctor of a left adjoint pair $\& \colon P_1 \times L_2 \to L_3$ in a similar way

[2]Note that, the irredundant finite $\vee$-decomposition of an join-irreducible element $p \in \mathcal{J}(L_1)$ is the proper element p, and therefore, $\&_y(p) = g_y(p)$ for all $p \in \mathcal{J}(L_1)$.

to the one given in Theorem 24. Consequently, we obtain analogous results to Theorem 24 and Proposition 25 related to left adjoint pairs.

Now, we will study the case of adjoint triples. Specifically, we will present how to define the conjunctor of an adjoint triple, by using the join-irreducible elements of distributive complete lattices satisfying the descending chain condition.

Theorem 26. *Let $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ be two distributive complete lattices satisfying the descending chain condition, $(L_3, \preceq_3)$ a complete lattice, $\mathcal{J}(L_1)$ and $\mathcal{J}(L_2)$ the sets of join-irreducible elements of L_1 and L_2, respectively. For all $x \in L_1$ and $y \in L_2$, we define the operator $\&\colon L_1 \times L_2 \to L_3$ as follows:*

$$x \,\&\, y = \begin{cases} \bot_3 & \textit{if} \quad x = \bot_1 \quad \textit{or} \quad y = \bot_2 \\ \displaystyle\bigvee_{p \in J(x)} \bigvee_{q \in J(y)} g(p, q) & \textit{otherwise} \end{cases} \tag{5}$$

where $J(x)$ and $J(y)$ denote the sets of join-irreducible elements of the irredundant finite $\vee$-decompositions of the elements x and y, respectively, and $g\colon \mathcal{J}(L_1) \times \mathcal{J}(L_2) \to L_3$ is an order-preserving mapping on both arguments[3]*. Then, the operator $\&$ is the conjunctor of an adjoint triple.*

Proof. To begin with, it is convenient to highlight that the mapping $\&$ is well-defined since, each element $x \in L_1$ and $y \in L_2$ has a unique irredundant finite $\vee$-decomposition. Now, we will see that the operator $\&$ is actually the conjunctor of an adjoint triple.

From a mapping $g\colon \mathcal{J}(L_1) \times \mathcal{J}(L_2) \to L_3$, which is order-preserving on both arguments, we can define the mappings $g_y\colon \mathcal{J}(L_1) \to L_3$ and $g_x\colon \mathcal{J}(L_2) \to L_3$ as $g_y(p) = \bigvee_{q \in J(y)} g(p, q)$ and $g_x(q) = \bigvee_{p \in J(x)} g(p, q)$, respectively, for each $x \in L_1 \setminus \{\bot_1\}$ and $y \in L_2 \setminus \{\bot_2\}$. Notice that, g_y and g_x are also order-preserving mappings. In addition, for each $x \in L_1 \setminus \{\bot_1\}$ and $y \in L_2 \setminus \{\bot_2\}$, we can define the conjunctor operators $\&_y\colon L_1 \to L_3$ and ${}_x\&\colon L_2 \to L_3$ as follows:

$$\&_y(x) = \begin{cases} \bot_3 & \text{if} \quad x = \bot_1 \\ \displaystyle\bigvee_{p \in J(x)} g_y(p) & \text{otherwise} \end{cases}$$

[3]Note that, the irredundant finite $\vee$-decomposition of an join-irreducible element is the proper element, and therefore, $p \,\&\, q = g(p, q)$ for all $(p, q) \in \mathcal{J}(L_1) \times \mathcal{J}(L_2)$.

$$
{}_{x}\&(y) = \begin{cases} \bot_3 & \text{if} \quad y = \bot_2 \\ \displaystyle\bigvee_{q \in J(y)} g_x(q) & \text{otherwise} \end{cases}
$$

Since, for each $x \in L_1 \setminus \{\bot_1\}$ and $y \in L_2 \setminus \{\bot_2\}$, the following equalities are satisfied:

$$
\begin{aligned}
\bigvee_{p \in J(x)} g_y(p) &= \bigvee_{p \in J(x)} \bigvee_{q \in J(y)} g(p, q) \\
\bigvee_{q \in J(y)} g_x(q) &= \bigvee_{p \in J(x)} \bigvee_{q \in J(y)} g(p, q)
\end{aligned}
$$

we obtain that the conjunctor operator $\&$ given by Equation (5) satisfies the chain of equalities $x \,\&\, y = \&_y(x) = {}_x\&(y)$, for all $x \in L_1$ and $y \in L_2$. Applying Theorem 24 and its dual, we can ensure that $\&$ is the conjunctor of a right adjoint pair and the conjunctor of a left adjoint pair. Thus, $\&$ has two adjoint implications satisfying the adjoint property, that is, it is the conjunctor of an adjoint triple. □

The next example clarifies the mechanism developed to define conjunctors of adjoint triples.

Example 27. We will consider the two distributive complete lattices satisfying the descending chain condition $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and the complete lattice $(L_3, \preceq_3)$ displayed in Figure 3. From the Hasse diagrams given in Figure 3, it is easy to see that the set of join-irreducible elements of L_1 and L_2 are $\mathcal{J}(L_1) = \{x_1, x_2\}$ and $\mathcal{J}(L_2) = \{y_1, \top_2\}$, respectively.

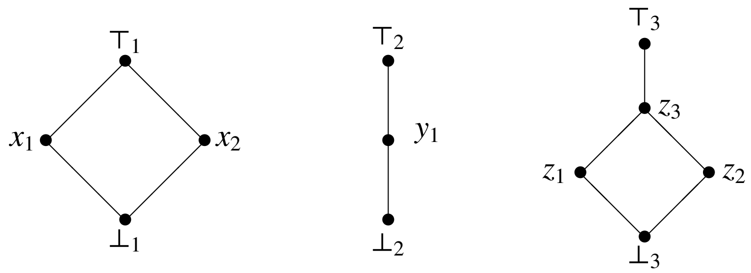

Figure 3: Lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L_3, \preceq_3)$ of Example 27.

For instance, we can obtain an adjoint triple from the order-preserving mapping $g\colon \mathcal{J}(L_1)\times\mathcal{J}(L_2) \to L_3$, defined as $g(x_1, y_1) = z_1$, $g(x_2, y_1) = z_2$, $g(x_1, \top_2) = g(x_2, \top_2) = z_3$. In order to compute the conjunctor $\&$ by using Equation (5), we need to consider the irredundant finite $\vee$-decompositions of the elements x_1, x_2,

$\top_1$, y_1 and $\top_2$. Taking into account that x_1 and y_1 can only be expressed as supremum of themselves, we have that $x_1 \,\&\, y_1 = g(x_1, y_1) = z_1$. A similar reasoning is carried out to compute $x_1 \,\&\, \top_2$, $x_2 \,\&\, y_1$ and $x_2 \,\&\, \top_2$. On the other hand, we have that the irredundant finite $\vee$-decomposition of the top element in L_1 is $\top_1 = \sup\{x_1, x_2\}$ and consequently:

$$\begin{array}{rcl} \top_1 \,\&\, y_1 & = & \sup\{g(x_1, y_1), g(x_2, y_1)\} = z_3 \\ \top_1 \,\&\, \top_2 & = & \sup\{g(x_1, \top_2), g(x_2, \top_2)\} = z_3 \end{array}$$

Applying Equation (5) and Proposition 16(3), we obtain the triple $(\&, \swarrow, \nwarrow)$ given in Table 3. We can guarantee that $(\&, \swarrow, \nwarrow)$ is an adjoint triple by Theorem 26.

$\&$	$\bot_2$	y_1	$\top_2$
$\bot_1$	$\bot_3$	$\bot_3$	$\bot_3$
x_1	$\bot_3$	z_1	z_3
x_2	$\bot_3$	z_2	z_3
$\top_1$	$\bot_3$	z_3	z_3

$\swarrow$	$\bot_2$	y_1	$\top_2$
$\bot_3$	$\top_1$	$\bot_1$	$\bot_1$
z_1	$\top_1$	x_1	$\bot_1$
z_2	$\top_1$	x_2	$\bot_1$
z_3	$\top_1$	$\top_1$	$\top_1$
$\top_3$	$\top_1$	$\top_1$	$\top_1$

$\nwarrow$	$\bot_1$	x_1	x_2	$\top_1$
$\bot_3$	$\top_2$	$\bot_2$	$\bot_2$	$\bot_2$
z_1	$\top_2$	y_1	$\bot_2$	$\bot_2$
z_2	$\top_2$	$\bot_2$	y_1	$\bot_2$
z_3	$\top_2$	$\top_2$	$\top_2$	$\top_2$
$\top_3$	$\top_2$	$\top_2$	$\top_2$	$\top_2$

Table 3: Definition of $(\&, \swarrow, \nwarrow)$ in Example 27.

Once again, we are interested in highlighting that the distributivity property is a necessary requirement in order to guarantee that the conjunctor operator given in Theorem 26 is well-defined. Assume that L_2 is the non-distributive lattice N_5 displayed in Figure 1. The set of join-irreducible elements of N_5 is given by $\mathcal{J}(N_5) = \{y_1, y_2, y_3\}$. Notice that, the element $\top_2$ has two different irredundant finite $\vee$-decompositions, $\top_2 = \sup\{y_1, y_2\}$ and $\top_2 = \sup\{y_2, y_3\}$.

For example, we will consider the mapping $g \colon \mathcal{J}(L_1) \times \mathcal{J}(L_2) \to L_3$ defined as $g(x_1, y_1) = g(x_1, y_2) = g(x_1, y_3) = z_1$, $g(x_2, y_1) = z_2$, $g(x_2, y_2) = z_3$ and $g(x_2, y_3) = \top_3$. Now, we will see that the operator $\&$ defined from g as in Equation (5) is not well-defined. Considering the irredundant finite $\vee$-decomposition $\top_2 = \sup\{y_1, y_2\}$, we obtain $x_2 \,\&\, \top_2 = \sup\{g(x_2, y_1), g(x_2, y_2)\} = \sup\{z_2, z_3\} = z_3$. On the other hand, considering the irredundant finite $\vee$-decomposition $\top_2 = \sup\{y_2, y_3\}$, we obtain $x_2 \,\&\, \top_2 = \sup\{g(x_2, y_2), g(x_2, y_3)\} = \sup\{z_2, \top_3\} = \top_3$. Since $x_2 \,\&\, \top_2$ takes different values depending on the chosen irredundant $\vee$-decomposition of $\top_2$, we conclude that the operator $\&$ obtained from Theorem 26 is not well-defined. $\square$

In the following, we will show that any conjunctor of an adjoint triple can be expressed as in Equation (5).

Proposition 28. *Let $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ be distributive complete lattices satisfying the descending chain condition and $(L_3, \preceq_3)$ a complete lattice. For any adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$, the conjunctor $\&$ satisfies Equation (5).*

Proof. Given an arbitrary adjoint triple $(\&, \swarrow, \nwarrow)$, applying Properties (3) and (4) of Proposition 15, we have that $x \& y = \perp_3$ when $x = \perp_1$ and/or $y = \perp_2$. Now, we will suppose that $x \neq \perp_1$ and $y \neq \perp_2$. Since $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are distributive complete lattices satisfying the descending chain condition, by Theorem 11, the elements x and y have a unique irredundant finite $\vee$-decomposition, namely $x = \bigvee_{p \in J(x)} p$ and $y = \bigvee_{q \in J(y)} q$, where $J(x)$ and $J(y)$ are the sets of join-irreducible elements in the decompositions. Taking into account Properties (6a) and (6c) of Proposition 15, we obtain that:

$$x \& y = \left(\bigvee_{p \in J(x)} p\right) \& \left(\bigvee_{q \in J(y)} q\right) = \bigvee_{p \in J(x)} \left(p \& \left(\bigvee_{q \in J(y)} q\right)\right) = \bigvee_{p \in J(x)} \bigvee_{q \in J(y)} (p \& q) = \bigvee_{p \in J(x)} \bigvee_{q \in J(y)} g(p, q)$$

where the mapping $g \colon \mathcal{J}(L_1) \times \mathcal{J}(L_2) \to L_3$ is defined as $g(p, q) = p \& q$, for all $p \in \mathcal{J}(L_1)$, $q \in \mathcal{J}(L_2)$. Clearly, g is order-preserving on both arguments. Hence, we can conclude that the conjunctor $\&$ satisfies Equation (5). $\square$

Theorem 26 and Proposition 28 enable us to ensure that the number of adjoint triples, which can be defined with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ finite distributive complete lattices verifying the descending chain condition and $(L_3, \preceq_3)$ a finite complete lattice, coincides with the number of order-preserving mappings on both arguments defined from $\mathcal{J}(L_1) \times \mathcal{J}(L_2)$ to L_3.

In this section, we have characterized the implications and conjunctors of Galois implications pairs, adjoint pairs and adjoint triples. From suitable algebraic stuctures (posets, distributive complete lattices, etc), we have seen that a considerable number of these residuated operators arises. In the following section, we will define an ordering relation on the set of all Galois implications pairs in order to prove that these residuated operators have the structure of a complete lattice. Similar results will be presented considering right adjoint pairs, left adjoint pairs and adjoint triples.

5. Algebraic structure of pairs and triples

In this section, we will begin introducing an ordering relation defined on the set of Galois implications pairs with respect to two complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and a poset $(P_3, \preceq_3)$. We will show that Galois implications pairs can be hierarchized giving rise to the structure of a complete lattice. Henceforth, the set of all Galois implications pairs, with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$, will be denoted as $\mathcal{I}$.

Proposition 29. *The pair $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ is a partially ordered set, where $\sqsubseteq_{\mathcal{I}}$ is the ordering relation defined as:*

$$(\swarrow^j, \nwarrow_j) \sqsubseteq_{\mathcal{I}} (\swarrow^k, \nwarrow_k) \qquad \textit{iff} \qquad z \swarrow^j y \preceq_1 z \swarrow^k y$$

for all $y \in L_2$, $z \in P_3$ and $(\swarrow^j, \nwarrow_j), (\swarrow^k, \nwarrow_k) \in \mathcal{I}$.

Proof. The proof straightforwardly follows from the reflexive, antisymmetric and transitive properties of $\preceq_1$. □

Notice that the inequalities $z \swarrow^j y \preceq_1 z \swarrow^k y$ and $z \nwarrow_j x \preceq_2 z \nwarrow_k x$ are equivalent, for all $x \in L_1$, $y \in L_2$, $z \in P_3$, as we show next. Suppose that $z \swarrow^j y \preceq_1 z \swarrow^k y$ and we will prove that the inequality $z \nwarrow_j x \preceq_2 z \nwarrow_k x$ holds. Applying Equivalence (2) to the trivial inequality $z \nwarrow_j x \preceq_2 z \nwarrow_j x$, we obtain that $x \preceq_1 z \swarrow^j (z \nwarrow_j x)$ holds, for all $x \in L_1$ and $z \in P_3$. Taking into account the hypothesis, we have that $x \preceq_1 z \swarrow^k (z \nwarrow_j x)$ and, by Equivalence (2), we conclude $z \nwarrow_j x \preceq_2 z \nwarrow_k x$. The counterpart, that is, $z \nwarrow_j x \preceq_2 z \nwarrow_k x$ implies $z \swarrow^j y \preceq_1 z \swarrow^k y$ can be deduced in an analogous way. Hence, the definition of the ordering relation introduced in Proposition 29 can be given equivalently from $\nwarrow_j$.

Two different Galois implications pairs $(\swarrow^j, \nwarrow_j)$ and $(\swarrow^k, \nwarrow_k)$ in $\mathcal{I}$ will be called incomparable when $(\swarrow^j, \nwarrow_j) \not\sqsubseteq_{\mathcal{I}} (\swarrow^k, \nwarrow_k)$ and $(\swarrow^k, \nwarrow_k) \not\sqsubseteq_{\mathcal{I}} (\swarrow^j, \nwarrow_j)$ and it will be denoted as $(\swarrow^j, \nwarrow_j) || (\swarrow^k, \nwarrow_k)$.

Theorem 30. *Let $\{(\swarrow^i, \nwarrow_i)\}_{i \in I} \subseteq \mathcal{I}$ be a non-empty arbitrary family of Galois implications pairs with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$. The mappings $\swarrow^{\inf} \colon P_3 \times L_2 \to L_1$ and $\nwarrow_{\inf} \colon P_3 \times L_1 \to L_2$, defined as:*

$$z \swarrow^{\inf} y = \bigwedge_{i \in I} \{z \swarrow^i y\}$$

$$z \nwarrow_{\inf} x = \bigwedge_{i \in I} \{z \nwarrow_i x\}$$

for all $x \in L_1$, $y \in L_2$, $z \in P_3$, form a Galois implications pair with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$. Furthermore, $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ is a meet-semilattice with maximum element.

Proof. Now, we will prove that $(\swarrow^{\inf}, \nwarrow_{\inf})$ is a Galois implications pair with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$ by means of Equivalence (2). We will suppose that the inequality $x \preceq_1 z \swarrow^{\inf} y$ is verified, being $x \in L_1, y \in L_2, z \in P_3$. As $x \preceq_1 \bigwedge_{i \in I}\{z \swarrow^i y\}$ then we have that $x \preceq_1 z \swarrow^i y$, for all $i \in I$. Taking into account that $(\swarrow^i, \nwarrow_i)$ is a Galois implications pair, the inequality $x \preceq_1 z \swarrow^i y$ is equivalent to $y \preceq_2 z \nwarrow_i x$, for all $i \in I$. From the infimum property, we obtain that $y \preceq_2 \bigwedge_{i \in I}\{z \nwarrow_i x\}$ and therefore, $y \preceq_2 z \nwarrow_{\inf} x$.

Following a similar reasoning to the previous one, we can prove the another implication and we can conclude that $(\swarrow^{\inf}, \nwarrow_{\inf})$ is a Galois implications pair with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$.

Now, we will demonstrate $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ is a meet-semilattice with a maximum element. We will see that the infimum of every non-empty family of Galois implications pairs of $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ exists. Clearly, $\swarrow^{\inf}$ is the infimum of $\{\swarrow^i\}_{i \in I}$ and $\nwarrow_{\inf}$ is the infimum of $\{\nwarrow_i\}_{i \in I}$, since the point-wise ordering between the implications have been considered (Proposition 29). Therefore, given a non-empty index set I and the family $\{(\swarrow^i, \nwarrow_i)\}_{i \in I} \subseteq \mathcal{I}$, we have that the Galois implications pair $(\swarrow^{\inf}, \nwarrow_{\inf})$ is the infimum of the family $\{(\swarrow^i, \nwarrow_i)\}_{i \in I}$ in $\mathcal{I}$.

Finally, it is easy to see that pair $(\swarrow^g, \nwarrow_g)$ defined as $z \swarrow^g y = \top_1$ and $z \nwarrow_g x = \top_2$, for all $x \in L_1$, $y \in L_2$, $z \in P_3$, is a Galois implications pair, which clearly is the maximum element of $\mathcal{I}$. As a consequence, we obtain that $(\swarrow, \nwarrow) \sqsubseteq_{\mathcal{I}} (\swarrow^g, \nwarrow_g)$, for all $(\swarrow, \nwarrow) \in \mathcal{I}$.

Thus, $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ is a meet-semilattice with maximum element. □

From Theorem 30 and Lemma 2, we can ensure that the set of all Galois implications pairs with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$ has the structure of a complete lattice.

Corollary 31. *$(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ is a complete lattice.*

Moreover, we have that the least pair of $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ is given by the mappings $\swarrow^l$, $\nwarrow_l$ defined for all $x \in L_1$, $y \in L_2$ and $z \in P_3$ as:

$$z \swarrow^l y = \begin{cases} \top_1 & \text{if } y = \bot_2 \\ \bot_1 & \text{otherwise} \end{cases} \qquad z \nwarrow_l x = \begin{cases} \top_2 & \text{if } x = \bot_1 \\ \bot_2 & \text{otherwise} \end{cases}$$

Given $(\swarrow, \nwarrow) \in \mathcal{I}$, by Proposition 23, we can define $\swarrow : P_3 \times L_2 \to L_1$ and $\nwarrow : P_3 \times L_1 \to L_2$ as in Equation (3), that is, $z \swarrow y = {}_z\swarrow (y)$ and $z \nwarrow x = {}^z\nwarrow (x)$, for all $z \in P_3$, $x \in L_1$ and $y \in L_2$. Note that, once we have fixed $z \in P_3$, the pair $({}_z\swarrow, {}^z\nwarrow)$ forms an antitone Galois connection. In [34], Theorem 1.6 shows that the set of Galois connections defined on posets has the structure of a complete lattice if and only if the posets are complete lattices. From this fact, Corollary 31 could be proved. Therefore, the main contribution of Theorem 30 consists in considering operators with two arguments, generalizing (classical and residuated) implications, providing a constructive way of defining the infimum operator of the complete lattice $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$. Furthermore, we have presented the definition of the maximum and minimum elements in $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$. Besides Theorem 1.6 in [34], there are other interesting results [34, Theorem 2.6 and Corollary 1.6] related to distributive lattices, which will be rewritten to $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ next.

Theorem 32. *$(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ is a completely distributive complete lattice if and only if $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are completely distributive complete lattices. In particular, if $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are finite distributive lattices, then $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ is distributive.*

Proof. The proof straightforwardly follows from Theorem 2.6 in [34]. $\square$

Hence, from the results above, we have introduced a constructive definition of the infimum in $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$. However, the determination of the supremum does not arise similarly, in general. Really, this is not a problem since, as it is usual in topped $\wedge$-structures, the supremum of arbitrary subsets of $\mathcal{I}$ can be computed as the infimum of the upper bounds of the subset in $\mathcal{I}$, although it would be very interesting in many aspects to obtain a constructive definition. The following proposition provides sufficient conditions in order to give an analytical definition of the supremum operator.

Proposition 33. *Let $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ be two distributive complete lattices satisfying the join-infinite distributive law, the ascending and descending chain conditions, $(P_3, \leq_3)$ be a poset and $\{(\swarrow^i, \nwarrow_i)\}_{i \in I} \subseteq \mathcal{I}$ be a non-empty arbitrary family of Galois implications pairs with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \leq_3)$. The mappings $\swarrow^{\sup} \colon P_3 \times L_2 \to L_1$ and $\nwarrow_{\sup} \colon P_3 \times L_1 \to L_2$, defined as:*

$$\begin{aligned} z \swarrow^{\sup} y &= \bigvee_{i \in I} \{z \swarrow^i y\} \\ z \nwarrow_{\sup} x &= \bigvee_{i \in I} \{z \nwarrow_i x\} \end{aligned}$$

for all $x \in L_1$, $y \in L_2$, $z \in P_3$, form a Galois implications pair $(\swarrow^{\sup}, \nwarrow_{\sup})$ with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \leq_3)$.

Proof. In order to prove that $(\swarrow^{\sup}, \nwarrow_{\sup})$ is a Galois implications pair, we firstly suppose that the inequality $x \preceq_1 z \swarrow^{\sup} y$ is verified being $x \in L_1$, $y \in L_2$, $z \in P_3$, that is, $x \preceq_1 \bigvee_{i \in I}\{z \swarrow^i y\}$. Applying Theorem 11, we can express $x \in L_1$ from its unique irredundant finite $\vee$-decomposition, that is, $x = \bigvee_{p \in J(x)} p$ where $J(x)$ is the set of join-irreducible elements of the unique irredundant finite $\vee$-decomposition of the element x. From this fact and taking into account the hypothesis, we obtain that the following chain of inequalities $p \preceq_1 x \preceq_1 \bigvee_{i \in I}\{z \swarrow^i y\}$ holds, for all $i \in I$ and $p \in J(x)$. Hence, by Lemma 13, for every $p \in J(x)$, there exists $i_p \in I$, such that $p \preceq_1 z \swarrow^{i_p} y$.

Since $(\swarrow^{i_p}, \nwarrow_{i_p})$ is a Galois implications pair, the inequality $p \preceq_1 z \swarrow^{i_p} y$ is equivalent to $y \preceq_2 z \nwarrow_{i_p} p$. Then, we can ensure that for every $p \in J(x)$, there exists $i_p \in I$, such that $y \preceq_2 z \nwarrow_{i_p} p$.

Therefore, we have the following chain of inequalities:

$$\begin{aligned}
y &\preceq_2 \bigwedge_{p \in J(x)} \{z \nwarrow_{i_p} p\} \\
&\preceq_2 \bigwedge_{p \in J(x)} \{z \nwarrow_{\sup} p\} \\
&= \bigwedge_{p \in J(x)} \left\{ \bigvee_{i \in I} \{z \nwarrow_i p\} \right\} \\
&\overset{(*)}{=} \bigvee_{i \in I} \left\{ \bigwedge_{p \in J(x)} \{z \nwarrow_i p\} \right\} \\
&= \bigvee_{i \in I} \left\{ z \nwarrow_i \left(\bigvee_{p \in J(x)} p \right) \right\} \\
&= \bigvee_{i \in I} \{z \nwarrow_i x\} \\
&= z \nwarrow_{\sup} x
\end{aligned}$$

where $(*)$ is given by the join-infinite distributive law of the lattice L_2. As a consequence, we obtain that $y \preceq_2 z \nwarrow_{\sup} x$. Following a similar reasoning, we prove that if $y \preceq_2 z \nwarrow_{\sup} x$, then $x \preceq_1 \bigvee_{i \in I}\{z \swarrow^i y\}$, for all $x \in L_1$, $y \in L_2$, $z \in P_3$. Thus, we have that $(\swarrow^{\sup}, \nwarrow_{\sup})$ is a Galois implication pair w.r.t $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$. □

As a consequence of the previous results, we have computed and hierarchized all Galois implications pairs associated with two complete lattices $(L_1, \preceq_1$

), $(L_2, \preceq_2)$ and a poset $(P_3, \preceq_3)$. This contribution gives us information related to the possible pairs that we can use in real applications, as well as the values that these pairs can take (according to the ordering relation $\sqsubseteq_{\mathcal{I}}$). Indeed, the results presented in this work, allow us to define various Galois implications pairs as different from each other as we need in our application. This fact drives, for example, to establish different degrees of preference between objects and attributes in frameworks such as multi-adjoint formal concept analysis, which can be very useful in real problems.

Note that, in the hypotheses of the proposition above, if $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are also completely distributive complete lattices then, by Theorem 2.6 presented in [34], besides to obtain a constructive definition of the supremum, by Theorem 32 we will also have that $(\mathcal{I}, \sqsubseteq_{\mathcal{I}})$ is a completely distributive complete lattice.

As a consequence of the previous results, we have computed and hierarchized all Galois implications pairs associated with two complete lattices $(L_1, \preceq_1$), $(L_2, \preceq_2)$ and a poset $(P_3, \preceq_3)$. This contribution gives us information related to the possible pairs that we can use in real applications, as well as the values that these pairs can take (according to the ordering relation $\sqsubseteq_{\mathcal{I}}$). Indeed, the results presented in this work, allow us to define various Galois implications pairs as different from each other as we need in our application. This fact drives, for example, to establish different degrees of preference between objects and attributes in frameworks such as multi-adjoint formal concept analysis, which can be very useful in real problems.

The next example shows that assuming a distributive lattice is a necessary condition in order to the pair $(\swarrow^{\sup}, \nwarrow_{\sup})$ defined in Proposition 33 be a Galois implications pair.

Example 34. Coming back to Example 22, we will consider the distributive complete lattice $(L_1, \preceq_1)$, the non-distributive complete lattice $(L_2, \preceq_2)$ and the poset $(P_3, \preceq_3)$, which are depicted in Figure 2 from left to right, respectively. From the Galois implications pairs $(\swarrow, \nwarrow)$ and $(\Swarrow, \Nwarrow)$ with respect to $(L_1, \preceq_1$ $)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$ displayed in Table 1 and Table 2, respectively, we will compute the pair $(\swarrow^{\sup}, \nwarrow_{\sup})$. By using Proposition 33, we obtain the pair $(\swarrow^{\sup}$ $, \nwarrow_{\sup})$ depicted in Table 4:

Now, we will show that $(\swarrow^{\sup}, \nwarrow_{\sup})$ is not a Galois implications pair with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \preceq_3)$. By Proposition 19, in order to see that $(\swarrow^{\sup}, \nwarrow_{\sup})$ is a Galois implications pair, we need to prove that $z \swarrow^{\sup} y = \max\{x \in L_1 \mid y \preceq_2 z \nwarrow_{\sup} x\}$, for all $y \in L_2$, $z \in P_3$, and that $\nwarrow_{\sup}$ is order-reversing on the second argument. Note that, $z_3 \swarrow^{\sup} y_1 = \sup\{x \in L_1 \mid y_1 \preceq_2$

$\swarrow^{\sup}$	$\bot_2$	y_1	y_2	y_3	$\top_2$
z_1	$\top_1$	x_2	$\bot_1$	$\bot_1$	$\bot_1$
z_2	$\top_1$	x_2	x_2	x_2	x_2
z_3	$\top_1$	$\top_1$	x_2	$\bot_1$	$\bot_1$
$\top_3$	$\top_1$	$\top_1$	$\bot_1$	x_2	$\bot_1$

$\nwarrow_{\sup}$	$\bot_1$	x_1	x_2	$\top_1$
z_1	$\top_2$	$\bot_2$	y_1	$\bot_2$
z_2	$\top_2$	$\bot_2$	$\top_2$	$\bot_2$
z_3	$\top_2$	y_1	$\top_2$	$\bot_2$
$\top_3$	$\top_2$	y_1	y_3	y_1

Table 4: Pair $(\swarrow^{\sup}, \nwarrow_{\sup})$ of Example 34.

$z_3 \nwarrow_{\sup} x\} = \sup\{\bot_1, x_1, x_2\} = \top_1$ and $\top_1 \notin \{x \in L_1 \mid y_1 \preceq_2 z_3 \nwarrow_{\sup} x\}$. Hence, we obtain that the supremum is not a maximum which leads us to conclude that $(\swarrow^{\sup}, \nwarrow_{\sup})$ is not a Galois implications pair.

Therefore, as $(L_2, \preceq_2)$ satisfies the descending chain condition, the distributivity property is a necessary condition in order to ensure that the operators defined in Proposition 33, $\swarrow^{\sup}$ and $\nwarrow_{\sup}$ form a Galois implications pair.

Notice that, considering a non-distributive complete lattice $(L_1, \preceq_1)$, a distributive complete lattice $(L_2, \preceq_2)$, both satisfying the descending chain condition, and a poset $(P_3, \leq_3)$, we can also find a family of Galois implications pairs with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(P_3, \leq_3)$ such that $\swarrow^{\sup}$ and $\nwarrow_{\sup}$ do not form a Galois implications pair. $\square$

Concerning the adjoint triples, we also need a similar study. First of all, an ordering relation will be defined on the set of all adjoint triples with respect to three complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \leq_3)$. This fact lead us to establish a hierarchy among adjoint triples. From now on, the set of all adjoint triples with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \leq_3)$ will be denoted as $\mathcal{T}$.

Proposition 35. *The pair $(\mathcal{T}, \sqsubseteq_{\mathcal{T}})$ is a partially ordered set, where $\sqsubseteq_{\mathcal{T}}$ is the ordering relation defined as:*

$$(\&_j, \swarrow^j, \nwarrow_j) \sqsubseteq_{\mathcal{T}} (\&_k, \swarrow^k, \nwarrow_k) \qquad \textit{iff} \qquad x \,\&_j\, y \leq_3 x \,\&_k\, y$$

for all $x \in L_1$, $y \in L_2$ and $(\&_j, \swarrow^j, \nwarrow_j), (\&_k, \swarrow^k, \nwarrow_k) \in \mathcal{T}$.

Proof. It is easy to see that the ordering relation $\sqsubseteq_{\mathcal{T}}$ satisfies the reflexive, antisymmetric and transitive properties from the properties of $\leq_3$. $\square$

If there exist two different adjoint triples $(\&_j, \swarrow^j, \nwarrow_j), (\&_k, \swarrow^k, \nwarrow_k) \in \mathcal{T}$ such that $(\&_j, \swarrow^j, \nwarrow_j) \not\sqsubseteq_{\mathcal{T}} (\&_k, \swarrow^k, \nwarrow_k)$ and $(\&_k, \swarrow^k, \nwarrow_k) \not\sqsubseteq_{\mathcal{T}} (\&_j, \swarrow^j, \nwarrow_j)$, then we will say that they are incomparable. In this case, we will write that $(\&_j, \swarrow^j, \nwarrow_j) \| (\&_k, \swarrow^k, \nwarrow_k)$.

Remark 36. It is also important to note that, by the monotonicity of the adjoint implications, the fixed ordering relation on the conjunctors of adjoint triples is the opposite to the ones given for their corresponding adjoint implications. Given two adjoint triples $(\&_j, \swarrow^j, \nwarrow_j), (\&_k, \swarrow^k, \nwarrow_k) \in \mathcal{T}$ such that $(\&_j, \swarrow^j, \nwarrow_j) \sqsubseteq_{\mathcal{T}} (\&_k, \swarrow^k, \nwarrow_k)$, then the inequalities $x \&_j y \preceq_3 x \&_k y$, $z \swarrow^k y \preceq_1 z \swarrow^j y$ and $z \nwarrow_k x \preceq_2 z \nwarrow_j x$ are equivalent, for all $x \in L_1$, $y \in L_2$ and $z \in L_3$. This fact can be deduced following a similar procedure to the one given after Proposition 29.

The previous remark justifies the definition of the adjoint triple introduced in the following result.

Theorem 37. *Let $\{(\&_i, \swarrow^i, \nwarrow_i)\}_{i \in I} \subseteq \mathcal{T}$ be a non-empty arbitrary family of adjoint triples with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$. The following mappings $\&_{\sup} \colon L_1 \times L_2 \to L_3$, $\swarrow^{\sup} \colon L_3 \times L_2 \to L_1$, $\nwarrow_{\sup} \colon L_3 \times L_1 \to L_2$ defined, for all $x \in L_1$, $y \in L_2$ and $z \in L_3$, as:*

$$
\begin{aligned}
x \&_{\sup} y &= \bigvee_{i \in I} \{x \&_i y\} \\
z \swarrow^{\sup} y &= \bigwedge_{i \in I} \{z \swarrow^i y\} \\
z \nwarrow_{\sup} x &= \bigwedge_{i \in I} \{z \nwarrow_i x\}
\end{aligned}
$$

form an adjoint triple $(\&_{\sup}, \swarrow^{\sup}, \nwarrow_{\sup})$ with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L_3, \preceq_3)$. Additionally, $(\mathcal{T}, \sqsubseteq_{\mathcal{T}})$ is a join-semilattice with minimum element.

Proof. First of all, we assume that the inequality $x \&_{\sup} y \preceq_3 z$ is verified, being $x \in L_1$, $y \in L_2$ and $z \in L_3$, that is, $\bigvee_{i \in I} \{x \&_i y\} \preceq_3 z$ holds. Applying the supremum property, we have that the inequality $x \&_i y \preceq_3 z$ is satisfied, for all $i \in I$. Taking into account that $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple, we obtain that $x \&_i y \preceq_3 z$ is equivalent to $x \preceq_1 z \swarrow^i y$, for all $i \in I$. By the infimum property, we have that $x \preceq_1 \bigwedge_{i \in I} \{z \swarrow^i y\} = z \swarrow^{\sup} y$ holds.

As the previous deductions are equivalences, if we suppose that $x \preceq_1 z \swarrow^{\sup} y = \bigwedge_{i \in I} \{z \swarrow^i y\}$, then we obtain that $\bigvee_{i \in I} \{x \&_i y\} \preceq_3 z$, that is $x \&_{\sup} y \preceq_3 z$.

The another equivalence $x \&_{\sup} y \preceq_3 z$ if and only if $y \preceq_2 z \nwarrow_{\sup} x$ can be proved in a similar way. Hence, $(\&_{\sup}, \swarrow^{\sup}, \nwarrow_{\sup})$ is an adjoint triple w.r.t $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$.

Now, we will demonstrate $(\mathcal{T}, \sqsubseteq_{\mathcal{T}})$ is a join-semilattice with a minimum element. First of all, we will see that the supremum of every non-empty family of adjoint triples of $(\mathcal{T}, \sqsubseteq_{\mathcal{T}})$ exists. Clearly, $\&_{\sup}$ is the supremum of $\{\&_i\}_{i \in I}$, since the point-wise ordering between the conjunctors have been considered (Proposition 35). Therefore, given a non-empty index set I and the family $\{(\&_i, \swarrow^i, \nwarrow_i)\}_{i\in I} \subseteq \mathcal{T}$, we have that $(\&_{\sup}, \Swarrow^{\sup}, \Nwarrow_{\sup})$ is the supremum of the family $\{(\&_i, \swarrow^i, \nwarrow_i)\}_{i\in I}$ in $\mathcal{T}$.

Finally, we clearly have that the triple $(\&_l, \swarrow^l, \nwarrow_l)$ whose operators are defined as $x \&_l y = \bot_3$, $z \swarrow^l y = \top_1$ and $z \nwarrow_l x = \top_2$, for all $x \in L_1, y \in L_2, z \in L_3$, is an adjoint triple and it is the minimum element of $\mathcal{T}$.

Consequently, $(\mathcal{T}, \sqsubseteq_{\mathcal{T}})$ is a join-semilattice with minimum element. □

Applying Theorem 37 and Lemma 2, we can conclude that the set of all adjoint triples with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$ has the structure of a complete lattice.

Corollary 38. *$(\mathcal{T}, \sqsubseteq_{\mathcal{T}})$ is a complete lattice.*

Note that, the adjoint implications with the greatest values appear at the bot of this complete lattice, unlike what happens with the complete lattice composed by Galois implications pairs. This fact is due to the considered ordering (Remark 36).

As it is usual in $\vee$-structures with a minimum element, the infimum of a family of adjoint triples needs to be computed as the supremum of the lower bounds of the family. The following result is analogous to Proposition 33 with respect to the infimum of $(\mathcal{T}, \sqsubseteq_{\mathcal{T}})$.

Proposition 39. *Let $\{(\&_i, \swarrow^i, \nwarrow_i)\}_{i\in I} \subseteq \mathcal{T}$ a non-empty arbitrary family of adjoint triples with respect to three distributive complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$ satisfying the ascending and descending chain conditions, the join-infinite distributive law in $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and the meet-infinite distributive law in $(L_3, \preceq_3)$. The following mappings $\&_{\inf} \colon L_1 \times L_2 \to L_3$, $\Swarrow^{\inf} \colon L_3 \times L_2 \to L_1$, $\Nwarrow_{\inf} \colon L_3 \times L_1 \to L_2$ defined, for all $x \in L_1$, $y \in L_2$ and $z \in L_3$, as:*

$$\begin{aligned} x \&_{\inf} y &= \bigwedge_{i\in I}\{x \&_i y\} \\ z \Swarrow^{\inf} y &= \bigvee_{i\in I}\{z \swarrow^i y\} \\ z \Nwarrow_{\inf} x &= \bigvee_{i\in I}\{z \nwarrow_i x\} \end{aligned}$$

form an adjoint triple $(\&_{\inf}, \Swarrow^{\inf}, \Nwarrow_{\inf})$ with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$.

Proof. First of all, we assume that the inequality $x \&_{\inf} y \preceq_3 z$ is verified, being $x \in L_1$, $y \in L_2$ and $z \in L_3$, that is, $\bigwedge_{i\in I}(x \&_i y) \preceq_3 z$ holds. Applying the dual of Theorem 11, we can express $z \in L_3$ from its unique irredundant finite $\wedge$-decomposition, that is, $z = \bigwedge_{q\in M(z)} q$ where $M(z)$ is the set of meet-irreducible elements of the unique irredundant finite $\wedge$-decomposition of the element z. From this fact and taking into account the hypothesis, we obtain that the following chain of inequalities $\bigwedge_{i\in I}(x \&_i y) \preceq_3 z \preceq_3 q$ holds, for all $q \in M(z)$. Hence, by the dual of Lemma 13, for every $q \in M(z)$, there exists $i_q \in I$, such that $x \&_{i_q} y \preceq_3 q$.

Since $(\&_{i_q}, \swarrow^{i_q}, \nwarrow_{i_q})$ is an adjoint triple, we have that the inequality $x \&_{i_q} y \preceq_3 q$ is equivalent to $y \preceq_2 q \nwarrow_{i_q} x$. Consequently, we can guarantee that for every $q \in M(z)$, there exists $i_q \in I$, such that $y \preceq_2 q \nwarrow_{i_q} x$. Therefore, we have the following chain of inequalities:

$$
\begin{aligned}
y &\preceq_2 \bigwedge_{q\in M(z)} \{q \nwarrow_{i_q} x\} \\
&\preceq_2 \bigwedge_{q\in M(z)} \{q \nwarrow_{\inf} x\} \\
&= \bigwedge_{q\in M(z)} \left\{ \bigvee_{i\in I} \{q \nwarrow_i x\} \right\} \\
&\overset{(*)}{=} \bigvee_{i\in I} \left\{ \bigwedge_{q\in M(z)} \{q \nwarrow_i x\} \right\} \\
&= \bigvee_{i\in I} \left\{ \left(\bigwedge_{q\in M(z)} q \right) \nwarrow_i x \right\} \\
&= \bigvee_{i\in I} \{z \nwarrow_i x\} \\
&= z \nwarrow_{\inf} x
\end{aligned}
$$

where $(*)$ is given by the join-infinite distributive law holds in the lattice $(L_2, \preceq_2)$.

Conversely, we suppose that $y \preceq_2 z \nwarrow_{\inf} x = \bigvee_{i\in I}(z \nwarrow_i x)$ holds, being $x \in L_1$, $y \in L_2$ and $z \in L_3$. We can apply Theorem 11 to express $y \in L_2$ from its unique irredundant finite $\vee$-decomposition obtaining that $y = \bigvee_{p\in J(y)} p$ where $J(y)$ is the set of join-irreducible elements of the decomposition. From this fact and taking into account the hypothesis, we obtain that the following chain of inequalities $p \preceq_2 y \preceq_2 \bigvee_{i\in I}\{z \nwarrow_i x\}$ holds, for all $i \in I$ and $p \in J(y)$. Hence, by

Lemma 13, for every $p \in J(y)$, there exists $i_p \in I$, such that $p \preceq_2 z \nwarrow_{i_p} x$.

Clearly, the inequality $p \preceq_2 z \nwarrow_{i_p} x$ is equivalent to $x \,\&_{i_p} p \preceq_3 z$, since $(\&_{i_p}, \swarrow^{i_p}, \nwarrow_{i_p})$ is an adjoint triple. Consequently, we can guarantee that for every $p \in J(y)$, there exists $i_p \in I$, such that $x \,\&_{i_p} p \preceq_3 z$, from which we have the following chain of inequalities:

$$\begin{array}{rcl} x \,\&_{\inf} y & = & \displaystyle\bigwedge_{i\in I}\{x \,\&_i\, y\} \\ & = & \displaystyle\bigwedge_{i\in I}\left\{x \,\&_i \left(\bigvee_{p\in J(y)} p\right)\right\} \\ & = & \displaystyle\bigwedge_{i\in I}\left\{\bigvee_{p\in J(y)}\{x \,\&_i\, p\}\right\} \\ & \overset{(*)}{=} & \displaystyle\bigvee_{p\in J(y)}\left\{\bigwedge_{i\in I}\{x \,\&_i\, p\}\right\} \\ & = & \displaystyle\bigvee_{p\in J(y)}\{x \,\&_{\inf} p\} \\ & \preceq_3 & \displaystyle\bigvee_{p\in J(y)}\{x \,\&_{i_p} p\} \\ & \preceq_3 & z \end{array}$$

where $(*)$ holds because $(L_3, \preceq_3)$ satisfies the meet-infinite distributive law.

Following a similar reasoning we prove the another equivalence $x \,\&_{\inf} y \preceq_3 z$ if and only if $x \preceq_1 z \swarrow^{\inf} y$ and we can conclude that $(\&_{\inf}, \swarrow^{\inf}, \nwarrow_{\inf})$ is an adjoint triple w.r.t $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$. □

The hypothesis of considering three distributive lattices is a necessary condition in order to guarantee that the triple $(\&_{\inf}, \swarrow^{\inf}, \nwarrow_{\inf})$ defined in Proposition 39 be an adjoint triple. This fact will be illustrated in the following example.

Example 40. Consider the distributive complete lattices $(L_1, \preceq_1)$, $(L_3, \preceq_3)$ and the non-distributive complete lattice $(L_2, \preceq_2)$ depicted in Figure 4 from left to right, respectively. Given the adjoint triples $(\&_1, \swarrow^1, \nwarrow_1)$ and $(\&_2, \swarrow^2, \nwarrow_2)$ defined in Table 5, we can compute the operators $\&_{\inf}$, $\swarrow^{\inf}$ and $\nwarrow_{\inf}$ which are defined in Proposition 39 and are also displayed in Table 5. By Proposition 16, if $(\&_{\inf}, \swarrow^{\inf}, \nwarrow_{\inf})$ is an adjoint triple with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L_3, \preceq_3)$, the following equalities must be verified:

$$\begin{array}{rcl} z \swarrow^{\inf} y & = & \max\{x \in L_1 \mid x \,\&_{\inf} y \preceq_3 z\} \\ z \nwarrow_{\inf} x & = & \max\{y \in L_2 \mid x \,\&_{\inf} y \preceq_3 z\} \end{array}$$

for all $x \in L_1$, $y \in L_2$ and $z \in L_3$. It is easy to check that these requirements are not satisfied by the operators $\&_{\mathrm{inf}}$, $\Swarrow^{\mathrm{inf}}$ and $\Nwarrow_{\mathrm{inf}}$. For example, considering the definition of $\Swarrow^{\mathrm{inf}}$ (given in Proposition 39) and Table 5, we have that $\bot_3 \Swarrow^{\mathrm{inf}} y_1 = (\bot_3 \swarrow^1 y_1) \vee (\bot_3 \swarrow^2 y_1) = \bot_1$. However, $\bot_3 \Swarrow^{\mathrm{inf}} y_1$ must be equal to $\max\{x \in L_1 \mid x \,\&_{\mathrm{inf}}\, y_1 \preceq_3 \bot_3\} = \max\{\bot_1, x_1\} = x_1$. As a consequence, we obtain that $(\&_{\mathrm{inf}}, \Swarrow^{\mathrm{inf}}, \Nwarrow_{\mathrm{inf}})$ is not an adjoint triple.

Taking into account that $(L_2, \preceq_2)$ satisfies the descending chain condition, we can conclude that the distributivity property is a necessary condition in order to ensure that the operators $\&_{\mathrm{inf}}$, $\Swarrow^{\mathrm{inf}}$ and $\Nwarrow_{\mathrm{inf}}$, defined in Proposition 39, form an adjoint triple.

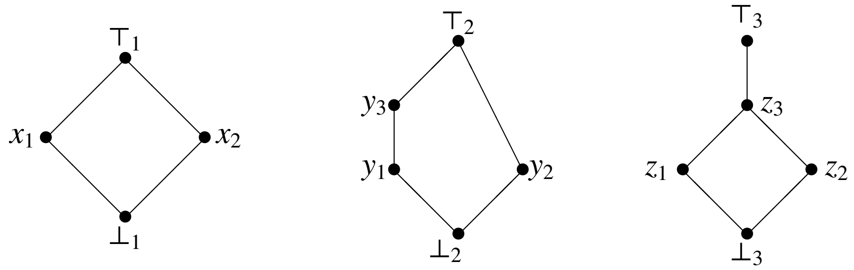

Figure 4: Lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L_3, \preceq_3)$ of Example 40.

It is convenient to mention that, when we consider either $(L_1, \preceq_1)$ or $(L_3, \preceq_3)$ is a non-distributive complete lattice, we can also find a family of adjoint triples with respect to $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L_3, \preceq_3)$ such that operators $\&_{\mathrm{inf}}$, $\Swarrow^{\mathrm{inf}}$ and $\Nwarrow_{\mathrm{inf}}$ do not form an adjoint triple. $\square$

Thus, a hierarchy among adjoint triples has also been introduced, which provides interesting advantages. Specifically, just like in the case of Galois implications pairs, this fact allows the user to define efficiently preferences among the objects and attributes on a database and, as a consequence, improve the capability of modeling the knowledge system in which these operators are applied.

It is worth noting that analogous results to the ones given to the set of all adjoint triples can be obtained to the set of all left (right, respectively) adjoint pairs, since we can define an ordering relation on the conjunctors of left (right, respectively) adjoint pairs as in Proposition 35. Therefore, we can guarantee that the set of all left (right, respectively) adjoint pairs also has structure of complete lattice.

$\&_1$	$\bot_2$	y_1	y_2	y_3	$\top_2$
$\bot_1$	$\bot_3$	$\bot_3$	$\bot_3$	$\bot_3$	$\bot_3$
x_1	$\bot_3$	z_1	z_1	z_1	z_1
x_2	$\bot_3$	z_2	z_2	z_2	z_2
$\top_1$	$\bot_3$	z_3	z_3	z_3	z_3

$\swarrow^1$	$\bot_2$	y_1	y_2	y_3	$\top_2$
$\bot_3$	$\top_1$	$\bot_1$	$\bot_1$	$\bot_1$	$\bot_1$
z_1	$\top_1$	x_1	x_1	x_1	x_1
z_2	$\top_1$	x_2	x_2	x_2	x_2
z_3	$\top_1$	$\top_1$	$\top_1$	$\top_1$	$\top_1$
$\top_3$	$\top_1$	$\top_1$	$\top_1$	$\top_1$	$\top_1$

$\nwarrow_1$	$\bot_1$	x_1	x_2	$\top_1$
$\bot_3$	$\top_2$	$\bot_2$	$\bot_2$	$\bot_2$
z_1	$\top_2$	$\top_2$	$\bot_2$	$\bot_2$
z_2	$\top_2$	$\bot_2$	$\top_2$	$\bot_2$
z_3	$\top_2$	$\top_2$	$\top_2$	$\top_2$
$\top_3$	$\top_2$	$\top_2$	$\top_2$	$\top_2$

$\&_2$	$\bot_2$	y_1	y_2	y_3	$\top_2$
$\bot_1$	$\bot_3$	$\bot_3$	$\bot_3$	$\bot_3$	$\bot_3$
x_1	$\bot_3$	z_2	z_2	z_2	z_2
x_2	$\bot_3$	z_3	z_3	z_3	z_3
$\top_1$	$\bot_3$	z_3	z_3	z_3	z_3

$\swarrow^2$	$\bot_2$	y_1	y_2	y_3	$\top_2$
$\bot_3$	$\top_1$	$\bot_1$	$\bot_1$	$\bot_1$	$\bot_1$
z_1	$\top_1$	$\bot_1$	$\bot_1$	$\bot_1$	$\bot_1$
z_2	$\top_1$	x_1	x_1	x_1	x_1
z_3	$\top_1$	$\top_1$	$\top_1$	$\top_1$	$\top_1$
$\top_3$	$\top_1$	$\top_1$	$\top_1$	$\top_1$	$\top_1$

$\nwarrow_2$	$\bot_1$	x_1	x_2	$\top_1$
$\bot_3$	$\top_2$	$\bot_2$	$\bot_2$	$\bot_2$
z_1	$\top_2$	$\bot_2$	$\bot_2$	$\bot_2$
z_2	$\top_2$	$\top_2$	$\bot_2$	$\bot_2$
z_3	$\top_2$	$\top_2$	$\top_2$	$\top_2$
$\top_3$	$\top_2$	$\top_2$	$\top_2$	$\top_2$

$\&_{\mathrm{inf}}$	$\bot_2$	y_1	y_2	y_3	$\top_2$
$\bot_1$	$\bot_3$	$\bot_3$	$\bot_3$	$\bot_3$	$\bot_3$
x_1	$\bot_3$	$\bot_3$	$\bot_3$	$\bot_3$	$\bot_3$
x_2	$\bot_3$	z_2	z_2	z_2	z_2
$\top_1$	$\bot_3$	z_3	z_3	z_3	z_3

$\Swarrow^{\mathrm{inf}}$	$\bot_2$	y_1	y_2	y_3	$\top_2$
$\bot_3$	$\top_1$	$\bot_1$	$\bot_1$	$\bot_1$	$\bot_1$
z_1	$\top_1$	x_1	x_1	x_1	x_1
z_2	$\top_1$	$\top_1$	$\top_1$	$\top_1$	$\top_1$
z_3	$\top_1$	$\top_1$	$\top_1$	$\top_1$	$\top_1$
$\top_3$	$\top_1$	$\top_1$	$\top_1$	$\top_1$	$\top_1$

$\Nwarrow_{\mathrm{inf}}$	$\bot_1$	x_1	x_2	$\top_1$
$\bot_3$	$\top_2$	$\bot_2$	$\bot_2$	$\bot_2$
z_1	$\top_2$	$\top_2$	$\bot_2$	$\bot_2$
z_2	$\top_2$	$\top_2$	$\top_2$	$\bot_2$
z_3	$\top_2$	$\top_2$	$\top_2$	$\top_2$
$\top_3$	$\top_2$	$\top_2$	$\top_2$	$\top_2$

Table 5: Definition of $(\&_1, \swarrow^1, \nwarrow_1)$, $(\&_2, \swarrow^2, \nwarrow_2)$ and $(\&_{\mathrm{inf}}, \Swarrow^{\mathrm{inf}}, \Nwarrow_{\mathrm{inf}})$ of Example 40.

6. Conclusions and further work

We have provided a characterization which define the conjunctors and the implications of Galois implications pairs, left/right adjoint pairs and adjoint triples, by using join-irreducible elements. The descending/ascending chain condition and the distributivity property of the complete lattices have played an important role in this characterization. We have defined an ordering relation on the set of all Galois implications pairs and consequently, we have established a hierarchy among them. In addition, we have proven that the set of all Galois implications pairs has structure of a complete lattice. An analogous study have been carried out with respect to adjoint triples and adjoint pairs. All these results, related to the algebraic structure of Galois implications pairs, adjoint triples and adjoint pairs, can be applied to general complete lattices such as infinite non-distributive lattices.

As a consequence, an efficient way to define general residuated operators either from examples or from data given in real cases, is presented. Therefore, given a set of initial values obtained from observations, we can build operators forming implications pairs, adjoint pairs and adjoint triples, to be considered in all the frameworks previously mentioned, such as in the fuzzy extensions of logic programming, formal concept analysis, rough theory, etc., and all these frameworks can be applied to the real problem associated with the observations.

For example, in the unit interval, since all the values are join-irreducible, except for 0, from any set of observed values, the monotonicity property is only required to define the pairs and triples given in Equations (3), (4), and (5). Therefore, these operators will always be computed, when the data have a monotonous character, which is the least expected property, if the data are associated with a conjunctor or an implication.

In the case of lattices, thanks to the obtained results, the construction is also efficient and simple. It is only needed to start from the values given by the observations and consider them as a subset of join-irreducible elements of the lattice. From this subset, the rest of the elements of the complete lattice can be defined, satisfying the distributive property and any other required condition. These procedures of computing implications pairs, adjoint pairs and adjoint triples from examples will be given in detail and applied to real cases in further extensions of this paper.

Furthermore, the theoretical advances achieved in this work will be very useful to study the algebraic structured formed by Galois implications pairs and adjoint triples whose adjoint negations [13] coincide with a given pair of weak negations [23], which is a future challenge that have emerged in the light of the obtained results.

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