Preferences in Discrete Multi-Adjoint Formal Concept Analysis*

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Abstract

Multi-adjoint concept lattice theory is a general fuzzy approach of formal concept analysis, which has diverse interesting properties. One of them is that it is possible to provide different degrees of preference among the set of objects/attributes. This paper studies a family of implications, based on the divisible discrete t-norms and in the Miller's law, which can be associated with a qualitative range of preference degrees to be considered in the applications by non-expert users of the FCA framework.

Keywords: Concept lattice; divisible discrete t-norm; fuzzy sets.

1. Introduction

Nowadays, preferences are omnipresent in most of the recommender systems around the world (Netflix, Youtube, Spotify, ...). These kinds of systems are personalized according to the user's preferences, that is, the systems and their computational mechanisms are put at the service of the

*Partially supported by the 2014–2020 ERDF Operational Programme in collaboration with the State Research Agency (AEI) in project PID2019-108991GB-I00, with the Ecological and Digital Transition Projects 2021 of the Ministry of Science and Innovation in project TED2021-129748B-I00, and with the Department of Economy, Knowledge, Business and University of the Regional Government of Andalusia in project FEDER-UCA18-108612, and by the European Cooperation in Science & Technology (COST) Action CA17124.

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user. In fact, personalization has been pointed out by Forbes as "the ultimate promise of the digital age" and several executives have established a direct impact on maximizing sales [23]. Therefore, the incorporation of preferences into knowledge-based systems is an important challenge today.

However, there exist several drawbacks when we ask someone to express his/her preference in terms of a real number. For instance, the same numerical value provided by two individuals may not have the same meaning, as Dubois discussed in [19]. In order to avoid this kind of situations, a limited qualitative scale that can be identically understood by several individuals, whose labels can be compared, needs to be considered. Indeed, it is well known that the memory plays an important role in the processes that humans perform for the acquisition of knowledge. Specifically, the working memory allows us to handle and to connect information in order to make decisions, solve certain problems and understand the language. This working memory is conceived as a temporary storage system of the information with a limited capacity and therefore, it is quickly overloaded. For example, George A. Miller stated that the number of units of information that a person can store in his/her working memory is seven plus or minus two. This statement is known in psychology as Miller's Law and is put forward in [40], which is one of the most cited papers in psychology.

In practice, Miller's Law can be interpreted as the number of degrees of preference that a person is able to consider online, e.g. for a set of objects to be evaluated. In fact, nowadays, when a film, a hotel, a restaurant, an athlete, etc., is evaluated, usually a scale of five values, and not ten or hundred, is considered. Indeed, there exist theories based on the consideration of less values, such as three-valued logic [31], Belnap's logic [5], three-way decisions [22, 46] or bipolarity (when positive, neutral and negative values are considered) [21].

Formal concept analysis (FCA) identifies conceptual structures in a data set that relates a set of attributes A and a set of objects B to each other by means of a binary relation $R \subseteq A \times B$. These pieces of information are called concepts and a hierarchy can be established on them providing an algebraic structure called concept lattice. From the concept lattice, a mathematical development for conceptual data analysis and processing of knowledge can be carried out. FCA is an actively researched mathematical tool from the theoretical [29, 36] and applicational [1, 3, 7, 12, 30, 42, 43, 45] point of view. Later, different fuzzy generalizations of FCA were introduced, as [2, 6, 8].

Fuzzy generalizations of the theory of FCA have been employed to solve several real problems, such as, multi-level data analysis, classification problems, analysis of solar power and weather open data, generation of linguistic description, rule learning and concept generation problems, among others. Some of the most recent works in which fuzzy FCA has been applied, can be found in [12, 13, 39, 41, 47]. For example, in [41] a new fuzzy rule-based classification model is proposed for fuzzy granular rule learning. In [3], this theory is considered to detect criminal patterns. In [27], fuzzy FCA is applied in the cloud environment to ensure the distribution of compute resources to the user. Fuzzy FCA is also applied in [44] to detect the feelings of the citizens toward technology, before and after the emergence of the Covid-19 pandemic. The sentiment analysis is also addresses from the perspective of fuzzy formal concept analysis in [25]. In [13], a new method for automatically generating linguistic descriptions by using residuated concept lattices is provided. In [12], multi adjoint formal concepts is applied to analyze solar power and weather open data in order to characterize the states of the sky and analyze the weather conditions, under which the energy production of photovoltaic panels is optimal.

The multi-adjoint concept lattice framework [34, 35, 37] can consider several adjoint triples in the definition of the concept-forming operations, providing interesting properties. One of these is the possibility of making up several clusters in the subsets of attributes and/or objects, which allows the consideration of different degrees of preference among them. The possibility of considering preferences in a (discrete) fuzzy concept lattice framework was presented in [11].

These preference degrees can be interpreted as the values of a membership function modeling a preference on the attributes and/or objects, following the semantics proposed by Zadeh, in which the values represent the intensity of preference in favor of a specific attribute/object [20]. This scale of membership values should be qualitative and small, as was argued above in order to be identically understood by several individuals (cognitively easier to grasp) and to allow for comparison [19]. Moreover, Dubois and Prade also noted the necessity of mapping them into a quantitative scale, although this must be well justified.

Following this idea, our main goal will consist of developing a fixed discrete and semantically complete structure considering at most seven linguistic labels to be mainly used in FCA Hence, since only the implications of the adjoint triples are considered in the definition of the concept-forming operators in FCA, we will be focused on the study of a family of implications in the algebraic structure. Moreover, the semantics of the different degrees of preference that the user can consider in this environment will be perfectly delimited and justified.

Therefore, this paper aims to break with the general trend of proposing more complex algebraic environments, which is interesting and may have an impact in the medium to long term, but needs a basic and applicable starting point in the short term. A discrete algebraic structure based on Miller's Law will be considered and a family of implications, appropriate to distinguish different degrees of a membership function of preference on the set of attributes or/and objects of a context, will be explored. This new approach will enable users without theoretical knowledge to use FCA while applying the usual semantic values of high preference, medium preference, etc., to a specific application. Therefore, the study developed in this paper will enrich the applications of FCA [3, 12, 13, 39, 41, 47], providing a formal framework to make the use of preferences in FCA more addresable for non-expert users.

This paper is structured as follows. In Section 2, we recall several necessary preliminary notions. A study of Fuzzy Formal Concept Analysis (FFCA) considering discrete operations is introduced in Section 3 and the advantages of the use of this kind of operations in FFCA are illustrated by means of a useful example in Section 4. Finally, the paper ends with several conclusions and prospects for future work.

2. Preliminaries

In order to keep the paper self-contained, the definitions and results needed throughout the paper will be recalled next.

2.1. Discrete triangular norms

The notion of triangular norm (t-norm) has been used in different fuzzy frameworks, such as in fuzzy logic, fuzzy relational equations, fuzzy rough sets, fuzzy formal concept analysis, etc. Its properties and extensions have been widely studied [16, 26, 28]. However, the non-countability of the unit interval $[0, 1]$ is a drawback instead of an advantage in different applications [17]. The following definition recalls the notion of a triangular norm on a finite chain.

Definition 1 ([24]). Let $C_n = \{x_1, \dots, x_n\}$ be a finite chain such that $x_1 < x_2 < \cdots < x_n$. A mapping $T\colon C_n \times C_n \to C_n$ is a *discrete t-norm* if it is commutative, associative, order-preserving and has x_n as neutral element, i.e.

$$T(x_i, x_n) = x_i, \quad \text{for all} \quad i \in \{1, 2, \ldots, n\}\,.$$

Two interesting properties related to idempotent elements of a discrete t-norm are introduced next, whose resolution is direct. The set of idempotent elements of T will be denoted as $\mathrm{Idem}(T)$.

Lemma 2. *Let $T, T^*\colon C_n \times C_n \to C_n$ be two discrete t-norms. If there exist $x_i \neq x_j \in C_n$, satisfying that $x_i \in Idem(T)$, $x_i \notin Idem(T^*)$ and $x_j \in Idem(T^*)$, $x_j \notin Idem(T)$, then T and T^* are incomparable, that is, $T \nleq T^*$ and $T^* \nleq T$.*

Lemma 3. *Given two discrete t-norms $T, T^*\colon C_n \times C_n \to C_n$ with residuated implications $\nwarrow_T, \nwarrow_{T^*}\colon C_n \times C_n \to C_n$. If $Idem(T) \subseteq Idem(T^*)$, we have that $\nwarrow_{T^*} \leq \nwarrow_T$.*

An interesting kind of t-norms are the divisible discrete t-norms characterized by Mayor and Torrens in [32].

Definition 4 ([24]). A discrete t-norm $T\colon C_n \times C_n \to C_n$ is called *divisible* if for any $i, j \in \{2, \ldots, n\}$ it holds that, if $T(x_i, x_j) = x_r$, then

$$T(x_{i-1}, x_j) = x_p \quad \text{and} \quad T(x_i, x_{j-1}) = x_q$$

with $r - 1 \leq p, q \leq r$.

Inspired by the Łukasiewicz t-norm on the real unit interval $[0, 1]$, a discrete divisible t-norm can be defined.

Lemma 5. *Given an integer number $k \in \mathbb{Z}$, a natural number $l \in \mathbb{N}$, and the index set $I = \{k, k+1, \ldots, k+l\}$, the operation $T_L\colon I \times I \to I$ defined by $T_L(i, j) = \max\{k, i + j - (k + l)\}$, for all $i, j \in I$, is a discrete divisible t-norm.*

From the characterization shown in [32], the following result arises.

Proposition 6. *Let T be a divisible discrete t-norm on a finite chain $C_n = \{x_1, \ldots, x_n\}$ and let $a_1 < a_2 < \cdots < a_k$ be the idempotent elements of T. Then, T is given by:*

$$T(x_i, x_j) \;=\; \begin{cases} x_{\max\{l, i+j-(l+1)\}} & \textit{if } x_i, x_j \in [a_l, a_{l+1}], \qquad l \in \{1, \ldots, k-1\} \\ x_{\min\{i,j\}} & \textit{otherwise} \end{cases}$$

for all $x_i, x_j \in C_n$.

2.2. Adjoint triples

Residuated t-norms or, equivalently, left-continuous t-norms, are also very useful operations in frameworks such as fuzzy relational equations, fuzzy logic, etc. This section directly introduces one of the most general operations with a residuum [9, 10], which generalizes left-continuous t-norms.

Definition 7. Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be three partially ordered sets (posets) and $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ be three mappings. We say that:

- $(\&, \swarrow)$ is a *right adjoint pair* with respect to P_1, P_2, P_3, if the equivalence
$$x \leq_1 z \swarrow y \quad \text{iff} \quad x \,\&\, y \leq_3 z \tag{1}$$
is satisfied, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$.

- $(\&, \nwarrow)$ is a *left adjoint pair* with respect to P_1, P_2, P_3, if the equivalence
$$x \leq_1 z \swarrow y \quad \text{iff} \quad y \leq_2 z \nwarrow x \tag{2}$$
holds, for all $x \in P_1$, $y \in P_2$ and $z \in P_3$.

Equivalences (1) and (2) are known as *adjointness properties*. The tuple $(\&, \swarrow, \nwarrow)$ is called *adjoint triple* with respect to P_1, P_2, P_3, if $(\&, \swarrow)$ and $(\&, \nwarrow)$ are left and right adjoint pairs to the corresponding posets, respectively.

Note that in the domain and codomain of the operations of an adjoint triple we could have three different posets, thus providing a more flexible language to a potential user. Furthermore, they satisfy the usual monotonicity properties although no boundary condition is required. In [38] more general examples of adjoint triples are given.

Clearly, the left-continuous t-norms, such as the Gödel, product and Łukasiewicz t-norms [28], together with their residuated implications form adjoint triples. Interesting non-commutative operators can also be considered, as the operator $\&\colon [0, 1] \times [0, 1] \to [0, 1]$ defined as:

$$\&(x, y) = x^2 y$$

for all $x, y \in [0, 1]$, which considers the variable evaluated on the right side more important than in the left side.[1] The residuated implications

[1] Note that $x^2 \leq x$, for all $x \in [0, 1]$.

$\swarrow : [0,1] \times [0,1] \to [0,1]$ and $\nwarrow : [0,1] \times [0,1] \to [0,1]$ are defined for all $x, y, z \in [0,1]$ as:

$$\begin{aligned} z \swarrow y &= \min\{1, \sqrt{z/y}\} \\ z \nwarrow x &= \min\{1, z/x\} \end{aligned}$$

The following property of adjoint triples shows how the adjoint implications can be defined from the conjunctor. In particular, this property proves the uniqueness of the implications associated with an adjoint conjunctor, which will be used later.

Proposition 8 ([10]). *Given three lattices* $(L_1, \preceq_1)$, $(L_2, \preceq_2)$, $(L_3, \preceq_3)$ *and an adjoint triple* $(\&, \swarrow, \nwarrow)$ *w.r.t.* L_1, L_2, L_3, *we have that*

$$\begin{aligned} z \swarrow y &= \max\{x' \in L_1 \mid x' \,\&\, y \preceq_3 z\} \\ z \nwarrow x &= \max\{y' \in L_2 \mid x \,\&\, y' \preceq_3 z\} \end{aligned}$$

for all $x \in L_1$, $y \in L_2$ *and* $z \in L_3$.

Note that if the conjunctor of an adjoint triple is commutative, then the corresponding residuated implications coincide [10].

Other interesting examples of left adjoint pairs, whose conjunctors also are commutative, are given in the Łukasiewicz family.

Definition 9. Given $\alpha \in [0,1]$, the operations $\&_\alpha, \nwarrow_\alpha : [0,1] \times [0,1] \to [0,1]$, defined by

$$x \,\&_\alpha\, y = \sqrt[1+\alpha]{\max\{0, x^{1+\alpha} + y^{1+\alpha} - 1\}}$$

$$z \nwarrow_\alpha x = \sqrt[1+\alpha]{\min\{1, 1 + z^{1+\alpha} - x^{1+\alpha}\}}$$

for all $x, y, z \in [0,1]$, form the left adjoint pair $(\&_\alpha, \nwarrow_\alpha)$. The set $\{(\&_\alpha, \nwarrow_\alpha)\}_{\alpha \in [0,1]}$ will be called *Łukasiewicz family.*

Adjoint triples are the basic operations of a remarkable mathematical tool for analyzing relational databases and representing conceptual knowledge in a (fuzzy) formal way, which is known as *multi-adjoint concept lattice* [37].

2.3. Multi-adjoint concept lattices

In formal concept analysis, the concepts are the basic units of information obtained from a given database. The operations used in order to compute these minimal pieces of information are called concept-forming operations and adjoint triples are considered in the definition of these fundamental operations. In this framework, we need to consider that $(P_1, \leq_1)$ and $(P_2, \leq_2)$ in Definition 7 are complete lattices [37]. The notion of multi-adjoint frame and context are given below.

Definition 10 ([37]). A *multi-adjoint frame* $\mathcal{L}$ is a tuple

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

where $(L_1, \preceq_1)$ and $(L_2, \preceq_2)$ are complete lattices, $(P, \leq)$ is a poset and, for each $i \in \{1, \dots, n\}$, $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple w.r.t. L_1, L_2, P. Multi-adjoint frames are denoted as $(L_1, L_2, P, \&_1, \dots, \&_n)$.

Definition 11 ([37]). Let $(L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame, a *context* is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon A \times B \to \{1, \dots, n\}$ is a mapping that associates any element in $A \times B$ with some particular adjoint triple in the frame.

Given a multi-adjoint frame and a context for that frame, the concept-forming operations are denoted[2] as ${}^{\uparrow_\sigma}\colon L_2^B \longrightarrow L_1^A$ and ${}^{\downarrow^\sigma}\colon L_1^A \longrightarrow L_2^B$ and are defined, for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, as

$$g^{\uparrow_\sigma}(a) = \inf\{R(a, b) \swarrow^{\sigma(a,b)} g(b) \mid b \in B\} \tag{3}$$
$$f^{\downarrow^\sigma}(b) = \inf\{R(a, b) \nwarrow_{\sigma(a,b)} f(a) \mid a \in A\} \tag{4}$$

Notice that these operators depend on the selection of implications of the adjoint triples offered by the mapping σ [18]. As in the classical case, these concept-forming operations form a Galois connection [37], and the notion of concept is defined as usual: a *multi-adjoint concept* is a pair $\langle g, f\rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^{\uparrow_\sigma} = f$ and $f^{\downarrow^\sigma} = g$; where $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ is the Galois connection defined above.

[2] L_2^B and L_1^A denote the set of fuzzy subsets $g\colon B \to L_2$, $f\colon A \to L_1$, respectively.

Definition 12. The *multi-adjoint concept lattice* associated with a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and a context (A, B, R, σ) is the set

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\sigma} = f, f^{\downarrow^\sigma} = g\}$$

in which the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ if and only if $g_1 \preceq_2 g_2$ (or equivalently, $f_2 \preceq_1 f_1$).

From now on, in order to simplify the notation, we will write $^\uparrow$ and $^\downarrow$ instead of $^{\uparrow_\sigma}$ and $^{\downarrow^\sigma}$, respectively.

As we commented previously, one important feature of this FCA framework is that different degrees of preference on the objects and attributes can be established throughout the mapping σ, as it was shown in [37] and this paper will continue studying. Indeed, the operator considered in the definition of the concept-forming operators are the residuated implications (Expressions 3 and 4) [18], and so the ordering among them will give the possibility of defining different degrees of preference. Next, a small example will be presented in order to illustrate this feature.

Example 13. In a factory, it is required to select the best worker for a specific task. Specifically, we have two workers $B = \{b_1, b_2\}$, two capacities $A = \{a_1, a_2\}$, and the degrees of expertise in each capacity, which is represented by the relation R given in Table 1.

Table 1: Relation R of Example 13.

R	b_1	b_2
a_1	0.5	0.5
a_2	0.7	0.3

Hence, if the manager needs a worker with at least 70% of expertise in both capacities, we can apply FCA to this small problem. For modeling the problem, we will consider the multi-adjoint frame $([0, 1], \leq, \&_G, \&_P, \&_L)$, where $\&_G, \&_P, \&_L$ are the Gödel, product and Łukasiewicz t-norms [28], and the context (A, B, R, σ), where σ is constantly the Gödel triple.

Therefore, for answering the question, we consider the fuzzy set $f\colon A \to [0, 1]$, defined as $f(a_1) = 0.7$, $f(a_2) = 0.7$, and we compute the following values:

$$
\begin{aligned}
f^{\downarrow^{\sigma}}(b_1) &= \inf\{R(a_1,b_1) \nwarrow_{\mathrm{G}} f(a_1), R(a_2,b_1) \nwarrow_{\mathrm{G}} f(a_2)\} \\
&= \inf\{0.5 \nwarrow_{\mathrm{G}} 0.7, 0.5 \nwarrow_{\mathrm{G}} 0.7\} \\
&= \inf\{0.5, 0.5\} = 0.5 \\
f^{\downarrow^{\sigma}}(b_2) &= \inf\{R(a_1,b_2) \nwarrow_{\mathrm{G}} f(a_1), R(a_2,b_2) \nwarrow_{\mathrm{G}} f(a_2)\} \\
&= \inf\{0.7 \nwarrow_{\mathrm{G}} 0.7, 0.3 \nwarrow_{\mathrm{G}} 0.7\} \\
&= \inf\{1, 0.3\} = 0.3
\end{aligned}
$$

As a consequence, the best worker satisfying the requirements is b_1. Now, if the manager would *prefer* worker b_2, because for example he/she has some recommendation letter or good references, the only way for obtaining a greater value for it, with no change in the context is to modify the implication operator. We can consider an implication different from the Gödel one, which gives greater values, that is, an implication $\nwarrow$, such that, $z \nwarrow_{\mathrm{G}} y \leq z \nwarrow y$, for all $z, y \in [0,1]$. For example, we could consider the product implication $\nwarrow_{\mathrm{P}}$, which satisfies $\nwarrow_{\mathrm{G}} \leq \nwarrow_{\mathrm{P}}$. In this case, the mapping $\sigma_1 \colon A \times B \to \{\mathrm{G}, \mathrm{P}, \text{Ł}\}$ is defined as $\sigma_1(a, b_1) = \mathrm{G}$ and $\sigma_1(a, b_2) = \mathrm{P}$, for all $a \in A$.

Therefore, we obtain that

$$
\begin{aligned}
f^{\downarrow^{\sigma_1}}(b_1) &= \inf\{R(a_1,b_1) \nwarrow_{\mathrm{G}} f(a_1), R(a_2,b_1) \nwarrow_{\mathrm{G}} f(a_2)\} \\
&= \inf\{0.5 \nwarrow_{\mathrm{G}} 0.7, 0.5 \nwarrow_{\mathrm{G}} 0.7\} \\
&= \inf\{0.5, 0.5\} = 0.5 \\
f^{\downarrow^{\sigma_1}}(b_2) &= \inf\{R(a_1,b_2) \nwarrow_{\mathrm{P}} f(a_1), R(a_2,b_2) \nwarrow_{\mathrm{P}} f(a_2)\} \\
&= \inf\{0.7 \nwarrow_{\mathrm{P}} 0.7, 0.3 \nwarrow_{\mathrm{P}} 0.7\} \\
&= \inf\{1, 0.3/0.7\} = 3/7
\end{aligned}
$$

In this case, we also have that $f^{\downarrow^{\sigma_1}}(b_2) = 3/7 \leq 0.5 = f^{\downarrow^{\sigma_1}}(b_1)$. This means that the manager can prefer another worker, a greater value is obtained, but this is not enough to be selected, that is, the user gives a preference, but not a commitment. This point of view is very interesting, since we continue considering both workers instead of removing some of them.

Finally, if the manager has a *strong preference* with respect to worker b_2, then we can chose the Łukasiewicz implication Ł instead of the Gödel one, since $\nwarrow_{\mathrm{G}} \leq \nwarrow_{\mathrm{P}} \leq \nwarrow_{\mathrm{L}}$. Hence, we consider the mapping $\sigma_2 \colon A \times B \to \{\mathrm{G}, \mathrm{P}, \text{Ł}\}$ defined as $\sigma_2(a, b_1) = \mathrm{G}$ and $\sigma_2(a, b_2) = \text{Ł}$, for all $a \in A$.

Therefore, we obtain that

$$\begin{aligned}
f^{\downarrow^{\sigma_2}}(b_1) &= \inf\{R(a_1,b_1) \nwarrow_{\mathrm{G}} f(a_1), R(a_2,b_1) \nwarrow_{\mathrm{G}} f(a_2)\} \\
&= \inf\{0.5 \nwarrow_{\mathrm{G}} 0.7, 0.5 \nwarrow_{\mathrm{G}} 0.7\} \\
&= \inf\{0.5, 0.5\} = 0.5 \\
f^{\downarrow^{\sigma_2}}(b_2) &= \inf\{R(a_1,b_2) \nwarrow_{\mathrm{Ł}} f(a_1), R(a_2,b_2) \nwarrow_{\mathrm{Ł}} f(a_2)\} \\
&= \inf\{0.7 \nwarrow_{\mathrm{Ł}} 0.7, 0.3 \nwarrow_{\mathrm{Ł}} 0.7\} \\
&= \inf\{1, 0.6\} = 0.6
\end{aligned}$$

As a consequence, in this strong preference case, worker b_2 will be selected.

Based on the Miller's Law, the following section will study a discrete multi-adjoint frame and a family of implications to be associated with different preference degrees that a user can consider. Anyway, the preferences could not affect the final result as we have shown above and we will see in the example of Section 4.

3. A theoretical study of a discrete multi-adjoint frame

This section will follow Miller's Law in order to study a family of implications and the relation to degrees of preference (semantic labels) so that a particular user can choose the most convenient implications for a specific problem in which multi-adjoint concept lattices will be used and degrees of preference among the attributes (or objects) will be considered. Since a finite scale is required, adjoint triples on finite domains will be considered preserving the main properties of discrete t-norms. The following section will present interesting discrete adjoint triples satisfying this goal.

3.1. Discrete adjoint triples

This section will start showing that discrete t-norms introduced in Lemma 5 have a residuated implication, which together form a left adjoint pair. These operations will be used later.

Lemma 14. *Let $k \in \mathbb{Z}$, $l \in \mathbb{N}$, and consider the index set $I = \{k, k+1, \dots, k+l\}$. The operations $\&_L, \nwarrow_L \colon I \times I \to I$ defined as:*

$$\begin{aligned}
i \,\&_L\, j &= \max\{k, i + j - (k+l)\} \\
h \nwarrow_L i &= \min\{k+l, h - i + k + l\}
\end{aligned}$$

for all $i, j, h \in I$, form a left adjoint pair with respect to I.

PROOF. The operators straightforwardly satisfy the adjointness property (Definition 7).

Now, proper discrete adjoint triples will be presented from t-norms considering regular partitions of $[0,1]$ [38]. For instance, $[0,1]_2 = \{0, 0.5, 1\}$ splits the unit interval into two pieces.

A discretization of a left-continuous t-norm $T\colon [0,1] \times [0,1] \to [0,1]$ is, e.g., the operator $T^*\colon [0,1]_n \times [0,1]_m \to [0,1]_k$, where $n, m, k \in \mathbb{N}$, and which is defined, for any $x \in [0,1]_n$ and $y \in [0,1]_m$, as:

$$T^*(x,y) = \frac{\lceil k \cdot T(x,y) \rceil}{k}$$

where $\lceil \_ \rceil$ is the ceiling function.

For this operation, the corresponding residuated implications $\swarrow^*\colon [0,1]_k \times [0,1]_m \to [0,1]_n$ and $\nwarrow_*\colon [0,1]_k \times [0,1]_n \to [0,1]_m$ are defined as:

$$z \swarrow^* y = \frac{\lfloor n \cdot (z \leftarrow y) \rfloor}{n} \qquad z \nwarrow_* x = \frac{\lfloor m \cdot (z \leftarrow x) \rfloor}{m}$$

where $\lfloor \_ \rfloor$ is the floor function and $\leftarrow$ is the residuated implication of the t-norm T. The triple $(T^*, \swarrow^*, \nwarrow_*)$ is an adjoint triple, where the operation T^* could be neither commutative nor associative.

The following example shows the particular case of the Łukasiewicz t-norm.

Example 15. A discretization of the Łukasiewicz t-norm is the operation $\&_{\mathrm{L}}^*\colon [0,1]_{20} \times [0,1]_8 \to [0,1]_{100}$ defined, for any $x \in [0,1]_{20}$ and $y \in [0,1]_8$ as:

$$x \,\&_{\mathrm{L}}^*\, y = \frac{\lceil 100 \cdot \max\{0, x + y - 1\} \rceil}{100}$$

whose residuated implications $\swarrow_{\mathrm{L}}^*\colon [0,1]_{100} \times [0,1]_8 \to [0,1]_{20}$, $\nwarrow_{\mathrm{L}}^*\colon [0,1]_{100} \times [0,1]_{20} \to [0,1]_8$ are defined as:

$$z \swarrow_{\mathrm{L}}^* y = \frac{\lfloor 20 \cdot \min\{1, 1 - y + z\} \rfloor}{20}$$

$$z \nwarrow_{\mathrm{L}}^* x = \frac{\lfloor 8 \cdot \min\{1, 1 - x + z\} \rfloor}{8}$$

Therefore, the triple $(\&_{\mathrm{L}}^*, \swarrow_{\mathrm{L}}^*, \nwarrow_{\mathrm{L}}^*)$ is an adjoint triple and the operation $\&_{\mathrm{L}}^*$ is neither commutative nor associative. Similar adjoint triples can be obtained from the Gödel and product t-norms.

This procedure can also be applied to the Łukasiewicz family introduced in Definition 9, obtaining a discrete adjoint triple family depending on $\alpha \in [0, 1]$.

3.2. The discrete multi-adjoint frame

As it was previously commented, the adjoint implications of the adjoint triples play a key role in the definition of the concept-forming operators. Therefore, the study of families of discrete implications is fundamental for introducing a discrete FFCA framework. There exist several possible families of implications, from a usual family of residuated implications (such as the ones obtained from the Łukasiewicz family or the divisible discrete t-norms) to an arbitrary subset of t-norms. However, the selection of the best family is not straightforward. This section will show that an interesting family is obtained from a subset of implications of divisible discrete t-norms. Nevertheless, not every subset of implications of divisible discrete t-norms provides a suitable family, as the following example shows.

Example 16. Let $C_4 = \{x_1, x_2, x_3, x_4\}$ be a finite chain such that $x_1 < x_2 < x_3 < x_4$ and consider the divisible discrete t-norms $T_i \colon C_4 \times C_4 \to C_4$, with $i \in \{1, 2, 3, 4\}$, defined in Table 2.

Table 2: Definitions of T_i

T_1	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_1	x_1	x_2
x_3	x_1	x_1	x_2	x_3
x_4	x_1	x_2	x_3	x_4

T_2	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_2	x_2
x_3	x_1	x_2	x_2	x_3
x_4	x_1	x_2	x_3	x_4

T_3	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_1	x_2	x_2
x_3	x_1	x_2	x_3	x_3
x_4	x_1	x_2	x_3	x_4

T_4	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_2	x_2
x_3	x_1	x_2	x_3	x_3
x_4	x_1	x_2	x_3	x_4

Note that all these t-norms have x_1 and x_4 as idempotent elements. Furthermore, x_2 is an idempotent element of T_2, x_3 is idempotent of T_3 and x_2, x_3 are idempotent elements of T_4.

The following result shows the uniqueness of these t-norms.

Proposition 17. *The operators T_1, T_2, T_3, and T_4 are the unique divisible discrete t-norms that can be defined on the finite chain $C_4 = \{x_1, x_2, x_3, x_4\}$.*

Proof. The proof follows from Proposition 6, since the divisible discrete t-norms are completely determined by the idempotent elements.

The following example presents the residuated implications of these t-norms.

Example 18. In the framework of Example 16, Table 3 shows the definitions of the corresponding residuated implications $\nwarrow_{T_i} : C_4 \times C_4 \to C_4$ with $i \in \{1, 2, 3, 4\}$.

Table 3: Definitions of $\nwarrow_{T_i}$

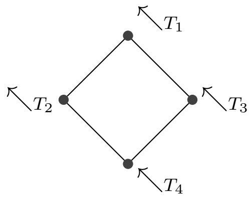
$\nwarrow_{T_1}$	x_1	x_2	x_3	x_4
x_1	x_4	x_3	x_2	x_1
x_2	x_4	x_4	x_3	x_2
x_3	x_4	x_4	x_4	x_3
x_4	x_4	x_4	x_4	x_4

$\nwarrow_{T_2}$	x_1	x_2	x_3	x_4
x_1	x_4	x_1	x_1	x_1
x_2	x_4	x_4	x_3	x_2
x_3	x_4	x_4	x_4	x_3
x_4	x_4	x_4	x_4	x_4

$\nwarrow_{T_3}$	x_1	x_2	x_3	x_4
x_1	x_4	x_2	x_1	x_1
x_2	x_4	x_4	x_2	x_2
x_3	x_4	x_4	x_4	x_3
x_4	x_4	x_4	x_4	x_4

$\nwarrow_{T_4}$	x_1	x_2	x_3	x_4
x_1	x_4	x_1	x_1	x_1
x_2	x_4	x_4	x_2	x_2
x_3	x_4	x_4	x_4	x_3
x_4	x_4	x_4	x_4	x_4

Considering the definitions of $\nwarrow_{T_i}$ and the point wise ordering, we have that $\nwarrow_{T_4} < \nwarrow_{T_2} < \nwarrow_{T_1}$ and $\nwarrow_{T_4} < \nwarrow_{T_3} < \nwarrow_{T_1}$, where $\nwarrow_{T_2}$ and $\nwarrow_{T_3}$ are incomparable operations according to Lemma 2, obtaining the following Hasse diagram:



Therefore, it is not enough to have a linear ordering among the t-norms in order to ensure a chain in the corresponding residuated implications. Hence, Lemma 3 should be considered for this goal. Moreover, since only four operators can be considered, in which the null and the neutral element are included, very few possibilities exist to associate the preference degrees of a user with them. Hence, more truth values (instead of C_4) would be necessary for a more proper modeling.

The standard number of values provided by Miller's Law is seven. Clearly, in this case, more divisible discrete t-norms can be defined. Indeed, when we consider chains with this number of values, by Lemma 3, we can ensure chains of implications with six different elements. Therefore, the user could consider six different "preference" degrees a priori.

Consequently, in order to obtain this chain of implications associated with divisible discrete t-norms, by Lemma 3, we should fix a chain of divisible discrete t-norms: $T_1, \dots, T_6$, such that $\mathrm{Idem}(T_i) \subseteq \mathrm{Idem}(T_{i+1})$, for all $i \in \{1, \dots, 5\}$.

We have two natural ways for defining this chain:[3] fixing the idempotent elements in the divisible discrete t-norms (1) from the smallest one, besides the minimum one, that is x_2, or (2) from the greatest one besides the maximum one, that is x_6. Due to the residuated implication $\swarrow$ of a left-continuous t-norm T satisfies:

$$z \swarrow y = \max\{x \in [0,1]_6 \mid T(x,y) \leq z\}$$

for all $y, z \in [0,1]_6$, the second option provides us with more different implications, because of the divisible discrete t-norms change from the greatest values if the idempotent elements are fixed from the greatest ones. For example, in Example 16, the chain corresponding to option (1) is $T_1 < T_2 < T_4$, where T_2 fixes as idempotent x_2 (besides x_1 and x_7). This sequence provides

[3]Notice that every divisible discrete t-norm has the minimum (x_1) and maximum (x_7) elements in the set as idempotent elements.

the chain of implications: $\nwarrow_{T_4} < \nwarrow_{T_2} < \nwarrow_{T_1}$, where the differences among the implications $\nwarrow_{T_4}$ and $\nwarrow_{T_2}$ are a single position (see Table 3), and between $\nwarrow_{T_2}$ and $\nwarrow_{T_1}$ there are two differences.

On the other hand, the chain associated with option (2) is $T_1 < T_3 < T_4$, where T_3 fixes x_3 as idempotent element, and obtaining the chain: $\nwarrow_{T_4} < \nwarrow_{T_3} < \nwarrow_{T_1}$, where the differences among the implications $\nwarrow_{T_4}$ and $\nwarrow_{T_3}$ are a single position, and between $\nwarrow_{T_2}$ and $\nwarrow_{T_1}$ the values of three positions change. These differences are clearly more important when the lattice increases, such as, when we will consider C_7 instead of C_4, as we can see in Table 4, in which the differences between $\nwarrow_{T_1}$ and $\nwarrow_{T_2}$ are 15, but between $\nwarrow_{T_1}$ and $\nwarrow_{T_2'}$ are only 5. Notice that it is more important for representing different degrees of preference that the number of positions change than the values in each position.

Thus, the chain obtained from option (2) on the granular interval $[0,1]_6$, that is, fixing the idempotent elements in the divisible discrete t-norms from x_6 to x_2, will provide a proper set of implications to be studied as a possible family to represent different degrees of preferences in FCA.

In the next section, we will introduce two interesting properties of divisible discrete t-norms that will be useful in computations.

3.3. Properties of the residuated implications of divisible discrete t-norms

The idempotent elements of a divisible discrete t-norm are fundamental, indeed, they determine the behaviour of the t-norm, as Proposition 6 proved. Therefore, they will have a direct impact in the definition of the residuated implications. The following result shows how the idempotent elements of a divisible discrete t-norm determines its residuated implication.

Theorem 19. *Given a finite chain $C_n = \{x_1, x_2, \dots, x_n\}$ with $n \in \mathbb{N}$ such that $x_1 < x_2 < \cdots < x_n$ and a divisible discrete t-norm $T\colon C_n \times C_n \to C_n$ together with its residuated implication $\nwarrow_T\colon C_n \times C_n \to C_n$, we have that*

(a) If x_i is an idempotent element of T, with $i \in \{2, \dots, n\}$, then $x_{i-1} \nwarrow_T x_i = x_{i-1}$.

(b) If x_k and x_l are consecutive idempotent elements of T then the equality $x_i \nwarrow_T x_j = x_{i-j+l}$ holds, for all $(x_i, x_j) \in [x_k, x_l] \times [x_k, x_l]$, with $i + 1 \leq j$.

Proof. First of all, we will prove the first statement. By Proposition 8, we have that:

$$x_{i-1} \nwarrow_T x_i = \max\{x_h \in C_n \mid T(x_i, x_h) \leq x_{i-1}\}$$

Table 4: Example of t-norms and residuated implications in $[0,1]_6$

T_1	0	1/6	2/6	3/6	4/6	5/6	1
0	0	0	0	0	0	0	0
1/6	0	0	0	0	0	0	1/6
2/6	0	0	0	0	0	1/6	2/6
3/6	0	0	0	0	1/6	2/6	3/6
4/6	0	0	0	1/6	2/6	3/6	4/6
5/6	0	0	1/6	2/6	3/6	4/6	5/6
1	0	1/6	2/6	3/6	4/6	5/6	1

$\swarrow^{T_1}$	0	1/6	2/6	3/6	4/6	5/6	1
0	1	5/6	4/6	3/6	2/6	1/6	0
1/6	1	1	5/6	4/6	3/6	2/6	1/6
2/6	1	1	1	5/6	4/6	3/6	2/6
3/6	1	1	1	1	5/6	4/6	3/6
4/6	1	1	1	1	1	5/6	4/6
5/6	1	1	1	1	1	1	5/6
1	1	1	1	1	1	1	1

T_2	0	1/6	2/6	3/6	4/6	5/6	1
0	0	0	0	0	0	0	0
1/6	0	0	0	0	0	1/6	1/6
2/6	0	0	0	0	1/6	2/6	2/6
3/6	0	0	0	1/6	2/6	3/6	3/6
4/6	0	0	1/6	2/6	3/6	4/6	4/6
5/6	0	1/6	2/6	3/6	4/6	5/6	5/6
1	0	1/6	2/6	3/6	4/6	5/6	1

$\swarrow^{T_2}$	0	1/6	2/6	3/6	4/6	5/6	1
0	1	4/6	3/6	2/6	1/6	0	0
1/6	1	1	4/6	3/6	2/6	1/6	1/6
2/6	1	1	1	4/6	3/6	2/6	2/6
3/6	1	1	1	1	4/6	3/6	3/6
4/6	1	1	1	1	1	4/6	4/6
5/6	1	1	1	1	1	1	5/6
1	1	1	1	1	1	1	1

T_2'	0	1/6	2/6	3/6	4/6	5/6	1
0	0	0	0	0	0	0	0
1/6	0	1/6	1/6	1/6	1/6	1/6	1/6
2/6	0	1/6	1/6	1/6	1/6	1/6	2/6
3/6	0	1/6	1/6	1/6	1/6	2/6	3/6
4/6	0	1/6	1/6	1/6	2/6	3/6	4/6
5/6	0	1/6	1/6	2/6	3/6	4/6	5/6
1	0	1/6	2/6	3/6	4/6	5/6	1

$\swarrow^{T_2'}$	0	1/6	2/6	3/6	4/6	5/6	1
0	1	0	0	0	0	0	0
1/6	1	1	5/6	4/6	3/6	2/6	1/6
2/6	1	1	1	5/6	4/6	3/6	2/6
3/6	1	1	1	1	5/6	4/6	3/6
4/6	1	1	1	1	1	5/6	4/6
5/6	1	1	1	1	1	1	5/6
1	1	1	1	1	1	1	1

with $i \in \{2, \ldots, n\}$.

Since x_i is an idempotent element of T, the equality $T(x_i, x_i) = x_i$ holds. In addition, T is order-preserving in the second argument, therefore we have that the elements x_h of C_n, such that $T(x_i, x_h) \le x_{i-1}$, must satisfy that $x_h < x_i$. Let us see that the supremum of these elements is x_{i-1}.

As T is a divisible discrete t-norm and $T(x_i, x_i) = x_i$, applying Definition 4, we obtain that $T(x_i, x_{i-1}) = x_q$ with $i-1 \leq q \leq i$. Since T is a t-norm, we have that $T(x_i, x_{i-1}) \leq \min\{x_i, x_{i-1}\} = x_{i-1}$. Hence, $T(x_i, x_{i-1}) = x_{i-1}$. As a consequence,

$$x_{i-1} \nwarrow_T x_i = \max\{x_h \in C_n \mid T(x_i, x_h) \leq x_{i-1}\} = x_{i-1}$$

Now, the second statement will be proven. Given $x_i, x_j \in [x_k, x_l]$, by Proposition 8, we have that

$$x_i \nwarrow_T x_j = \max\{x_h \in C_n \mid T(x_j, x_h) \leq x_i\}\,.$$

We firstly prove that the elements $x_{h'} \in \{x_h \in C_n \mid T_i(x_j, x_h) \leq x_i\}$ satisfy $x_{h'} \leq x_l$. If we assume that $x_l \leq x_{h'}$, by the monotonicity of T, we obtain that

$$T(x_j, x_l) \leq T(x_j, x_{h'}) \leq x_i\,.$$

On the other hand, by Proposition 6, the equality $T(x_j, x_l) = x_{\max\{k, j+l-l\}} = x_j$ holds. Hence, as a consequence of the two previous inequalities, we obtain $x_j \leq x_i$ which contradicts the hypothesis $i+1 \leq j$. Therefore, we can conclude that $x_{h'} \leq x_l$, for all $x_{h'} \in \{x_h \in C_n \mid T(x_j, x_h) \leq x_i\}$.

Moreover, since $T(x_j, x_k) = x_k \leq x_i$, we have $x_k \in \{x_h \in C_n \mid T(x_j, x_h) \leq x_i\}$, and so

$$\begin{aligned}
x_i \nwarrow_T x_j &= \sup\{x_h \in C_n \mid T(x_j, x_h) \leq x_i\} \\
&= \sup\{x_h \in C_n \mid T(x_j, x_h) \leq x_i, x_k \leq x_h \leq x_l\} \\
&\stackrel{(1)}{=} \sup\{x_h \in C_n \mid x_{\max\{j+h-l, k\}} \leq x_i\} \\
&= x_{\sup\{h \in \{k, \ldots, l\} \mid \max\{j+h-l, k\} \leq i\}} \\
&\stackrel{(2)}{=} x_{i-j+l}
\end{aligned}$$

where (1) follows from Proposition 6 and $x_j, x_h \in [x_k, x_l]$, and (2) because $k \leq i-j+l < l$.

This result will be used in the following section in order to compute the discrete divisible t-norms defined on the granular interval with seven values, following Option (2) above.

3.4. Proposed family of implications and its relation to degrees of preference

As it was previously argued, a family of implications defined on the granular interval with seven values will be considered. Specifically, the

Table 5: Definitions of divisible discrete t-norms T_i on $[0,1]_6$

T_1	0	1/6	2/6	3/6	4/6	5/6	1
0	0	0	0	0	0	0	0
1/6	0	0	0	0	0	0	1/6
2/6	0	0	0	0	0	1/6	2/6
3/6	0	0	0	0	1/6	2/6	3/6
4/6	0	0	0	1/6	2/6	3/6	4/6
5/6	0	0	1/6	2/6	3/6	4/6	5/6
1	0	1/6	2/6	3/6	4/6	5/6	1

T_2	0	1/6	2/6	3/6	4/6	5/6	1
0	0	0	0	0	0	0	0
1/6	0	0	0	0	0	1/6	1/6
2/6	0	0	0	0	1/6	2/6	2/6
3/6	0	0	0	1/6	2/6	3/6	3/6
4/6	0	0	1/6	2/6	3/6	4/6	4/6
5/6	0	1/6	2/6	3/6	4/6	5/6	5/6
1	0	1/6	2/6	3/6	4/6	5/6	1

T_3	0	1/6	2/6	3/6	4/6	5/6	1
0	0	0	0	0	0	0	0
1/6	0	0	0	0	1/6	1/6	1/6
2/6	0	0	0	1/6	2/6	2/6	2/6
3/6	0	0	1/6	2/6	3/6	3/6	3/6
4/6	0	1/6	2/6	3/6	4/6	4/6	4/6
5/6	0	1/6	2/6	3/6	4/6	5/6	5/6
1	0	1/6	2/6	3/6	4/6	5/6	1

T_4	0	1/6	2/6	3/6	4/6	5/6	1
0	0	0	0	0	0	0	0
1/6	0	0	0	0	1/6	1/6	1/6
2/6	0	0	1/6	2/6	2/6	2/6	2/6
3/6	0	1/6	2/6	3/6	3/6	3/6	3/6
4/6	0	1/6	2/6	3/6	4/6	4/6	4/6
5/6	0	1/6	2/6	3/6	4/6	5/6	5/6
1	0	1/6	2/6	3/6	4/6	5/6	1

T_5	0	1/6	2/6	3/6	4/6	5/6	1
0	0	0	0	0	0	0	0
1/6	0	0	1/6	1/6	1/6	1/6	1/6
2/6	0	1/6	2/6	2/6	2/6	2/6	2/6
3/6	0	1/6	2/6	3/6	3/6	3/6	3/6
4/6	0	1/6	2/6	3/6	4/6	4/6	4/6
5/6	0	1/6	2/6	3/6	4/6	5/6	5/6
1	0	1/6	2/6	3/6	4/6	5/6	1

T_6	0	1/6	2/6	3/6	4/6	5/6	1
0	0	0	0	0	0	0	0
1/6	0	1/6	1/6	1/6	1/6	1/6	1/6
2/6	0	1/6	2/6	2/6	2/6	2/6	2/6
3/6	0	1/6	2/6	3/6	3/6	3/6	3/6
4/6	0	1/6	2/6	3/6	4/6	4/6	4/6
5/6	0	1/6	2/6	3/6	4/6	5/6	5/6
1	0	1/6	2/6	3/6	4/6	5/6	1

regular partition of the unit interval $[0,1]_6$ and the residuated implications of the t-norms $T_i\colon [0,1]_6 \times [0,1]_6 \to [0,1]_6$, with $i \in \{1,2,3,4,5,6\}$, which are defined in Tables 5 and 6, will be taken into account. It is easy to verify that the operations T_i are divisible discrete t-norms.

Note that all these t-norms have 0 and 1 as idempotent elements, and satisfy Lemma 3. Specifically, the sequence of idempotent elements is as

Table 6: Definitions of the residuated implications $\nwarrow_{T_i}$ on $[0,1]_6$

$\nwarrow_{T_1}$	0	1/6	2/6	3/6	4/6	5/6	1
0	1	5/6	4/6	3/6	2/6	1/6	0
1/6	1	1	5/6	4/6	3/6	2/6	1/6
2/6	1	1	1	5/6	4/6	3/6	2/6
3/6	1	1	1	1	5/6	4/6	3/6
4/6	1	1	1	1	1	5/6	4/6
5/6	1	1	1	1	1	1	5/6
1	1	1	1	1	1	1	1

$\nwarrow_{T_2}$	0	1/6	2/6	3/6	4/6	5/6	1
0	1	4/6	3/6	2/6	1/6	0	0
1/6	1	1	4/6	3/6	2/6	1/6	1/6
2/6	1	1	1	4/6	3/6	2/6	2/6
3/6	1	1	1	1	4/6	3/6	3/6
4/6	1	1	1	1	1	4/6	4/6
5/6	1	1	1	1	1	1	5/6
1	1	1	1	1	1	1	1

$\nwarrow_{T_3}$	0	1/6	2/6	3/6	4/6	5/6	1
0	1	3/6	2/6	1/6	0	0	0
1/6	1	1	3/6	2/6	1/6	1/6	1/6
2/6	1	1	1	3/6	2/6	2/6	2/6
3/6	1	1	1	1	3/6	3/6	3/6
4/6	1	1	1	1	1	4/6	4/6
5/6	1	1	1	1	1	1	5/6
1	1	1	1	1	1	1	1

$\nwarrow_{T_4}$	0	1/6	2/6	3/6	4/6	5/6	1
0	1	2/6	1/6	0	0	0	0
1/6	1	1	2/6	1/6	1/6	1/6	1/6
2/6	1	1	1	2/6	2/6	2/6	2/6
3/6	1	1	1	1	3/6	3/6	3/6
4/6	1	1	1	1	1	4/6	4/6
5/6	1	1	1	1	1	1	5/6
1	1	1	1	1	1	1	1

$\nwarrow_{T_5}$	0	1/6	2/6	3/6	4/6	5/6	1
0	1	1/6	0	0	0	0	0
1/6	1	1	1/6	1/6	1/6	1/6	1/6
2/6	1	1	1	2/6	2/6	2/6	2/6
3/6	1	1	1	1	3/6	3/6	3/6
4/6	1	1	1	1	1	4/6	4/6
5/6	1	1	1	1	1	1	5/6
1	1	1	1	1	1	1	1

$\nwarrow_{T_6}$	0	1/6	2/6	3/6	4/6	5/6	1
0	1	0	0	0	0	0	0
1/6	1	1	1/6	1/6	1/6	1/6	1/6
2/6	1	1	1	2/6	2/6	2/6	2/6
3/6	1	1	1	1	3/6	3/6	3/6
4/6	1	1	1	1	1	4/6	4/6
5/6	1	1	1	1	1	1	5/6
1	1	1	1	1	1	1	1

follows:

$$
\begin{aligned}
\mathrm{Idem}(T_1) &= \{0, 1\} \\
\mathrm{Idem}(T_2) &= \{0, 5/6, 1\}. \\
\mathrm{Idem}(T_3) &= \{0, 4/6, 5/6, 1\} \\
\mathrm{Idem}(T_4) &= \{0, 3/6, 4/6, 5/6, 1\} \\
\mathrm{Idem}(T_5) &= \{0, 2/6, 3/6, 4/6, 5/6, 1\} \\
\mathrm{Idem}(T_6) &= \{0, 1/6, 2/6, 3/6, 4/6, 5/6, 1\} = [0,1]_6
\end{aligned}
$$

Therefore, we can easily see that $T_1 < T_2 < T_3 < T_4 < T_5 < T_6$. Moreover, considering the corresponding residuated implications $\nwarrow_{T_i} : C \times C \to C$ with $i \in \{1, 2, 3, 4, 5, 6\}$, we have that

$$\nwarrow_{T_6} < \nwarrow_{T_5} < \nwarrow_{T_4} < \nwarrow_{T_3} < \nwarrow_{T_2} < \nwarrow_{T_1} \tag{5}$$

Next, we will show that the residuated implications $\nwarrow_{T_i}$ differ significantly and so, they can be considered to represent different degrees of preferences in FCA, as it will be shown in Table 9.

From now on, our goal consists of consistently mapping a small qualitative scale of degrees of preferences into adjoint implications, following the recommendation given in [19]:

> "A small qualitative scale is cognitively easier to grasp than a continuous value scale and thus more chance to the consensual"

To reach this goal, adjoint implications will be compared using two distance measures. Given two implications $\nwarrow_{T_i}$ and $\nwarrow_{T_j}$, with $i, j \in \{1, \dots, 6\}$, the Manhattan distance between $\nwarrow_{T_i}$ and $\nwarrow_{T_j}$ is defined as

$$D_1(\nwarrow_{T_i}, \nwarrow_{T_j}) = \sum_{0 \leq l < k \leq 6} |(z_l \nwarrow_{T_i} x_k) - (z_l \nwarrow_{T_j} x_k)|$$

and the Euclidean distance is defined as

$$D_2(\nwarrow_{T_i}, \nwarrow_{T_j}) = \sqrt{\sum_{0 \leq l < k \leq 6} ((z_l \nwarrow_{T_i} x_k) - (z_l \nwarrow_{T_j} x_k))^2}$$

Notice that the consideration of both distances is important. Manhattan distance measures how many 'steps' (of one unit degree -$1/6$) exist between two implications, whilst the Euclidean distance also takes into account whether in a position there is more than one step of difference. For example, in the implication represented in Table 4, we noted that the differences between $\nwarrow_{T_1}$ and $\nwarrow_{T_2}$ are 15 positions, that is, 15 increments in steps of $1/6$ from a value in the table of T_2 to a value in the same position in the table of T_1, e.g. from $(0 \nwarrow_{T_2} 1/6) = 4/6$ to $(0 \nwarrow_{T_1} 1/6) = 5/6$, we only have one increment. However, between $\nwarrow_{T_1}$ and $\nwarrow_{T_2'}$ only 5 position exist. Nevertheless, if we consider the Manhattan distance we have $D_1(\nwarrow_{T_1}, \nwarrow_{T_2}) = 15$, and $D_1(\nwarrow_{T_1}, \nwarrow_{T_2'}) = 15$, because the real number of steps are 15 in both cases, although in the first difference is distributed among more positions, which is naturally better for defining a family of implications for representing different degrees of preference in FCA.

Considering the Euclidean distance, the differences between both pairs of implications is clearer, because it shows some uniformity among the differences. We obtain that $D_2(\nwarrow_{T_1}, \nwarrow_{T_2}) = \sqrt{15/6^2}$, and $D_2(\nwarrow_{T_1}, \nwarrow_{T_2'}) = \sqrt{55/6^2}$, which show a better uniformity in the first comparison than in the second one, that is, the differences between $\nwarrow_{T_1}$, and $\nwarrow_{T_2}$ are divided into more positions than the ones given to $\nwarrow_{T_1}$, and $\nwarrow_{T_2'}$. Thus, the goal is to obtain a chain of implications, whose differences have a bigger number of steps (great Manhattan distance) in as many positions as possible (small Euclidean distance).

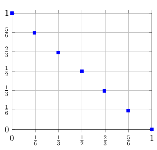
Table 7 shows the distances between two consecutive implications in the chain introduced in (5). From this table, a suitable difference among the implications holds, except with respect to the pairs T_4 and T_5, T_5 and T_6. Hence, the implication T_5 may be discarded.

Table 7: Manhattan distance D_1 and Euclidean distance D_2

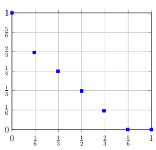
	D_1	D_2
$\lvert\nwarrow_{T_1} - \nwarrow_{T_2}\rvert$	2.500	0.645
$\lvert\nwarrow_{T_2} - \nwarrow_{T_3}\rvert$	1.666	0.527
$\lvert\nwarrow_{T_3} - \nwarrow_{T_4}\rvert$	1.000	0.408
$\lvert\nwarrow_{T_4} - \nwarrow_{T_5}\rvert$	0.500	0.288
$\lvert\nwarrow_{T_5} - \nwarrow_{T_6}\rvert$	0.166	0.166

The implications from $\nwarrow_{T_1}$ to $\nwarrow_{T_6}$ are depicted from left to right in the following graphics. Each of the rows corresponds to a fixed value of the consequent (z), which provides the mappings $z \nwarrow_{T_i} \_ \colon [0,1]_6 \to [0,1]_6$, for all $i \in \{1, \ldots, 6\}$. The values of the antecedent (x) are represented in the horizontal axis whereas the values corresponding to the implications (y) are given in the vertical axis. Specifically, if $z = 0$ is fixed in the six implications we obtain the six restricted mappings on $[0,1]_6$ (with only one argument) depicted below:

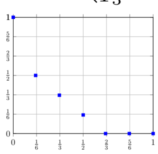
$0 \nwarrow_{T_1} \_$



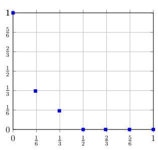
$0 \nwarrow_{T_2} \_$



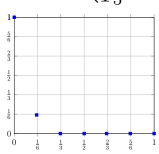
$0 \nwarrow_{T_3} \_$



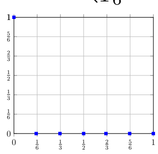
$0 \nwarrow_{T_4} \_$



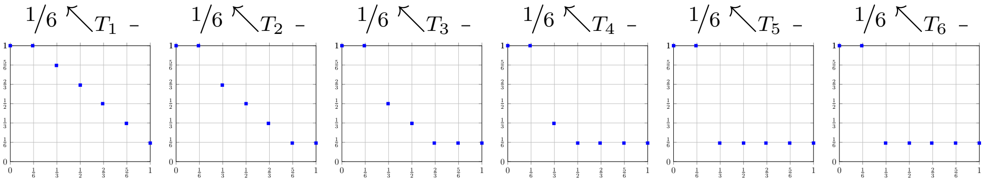
$0 \nwarrow_{T_5} \_$



$0 \nwarrow_{T_6} \_$

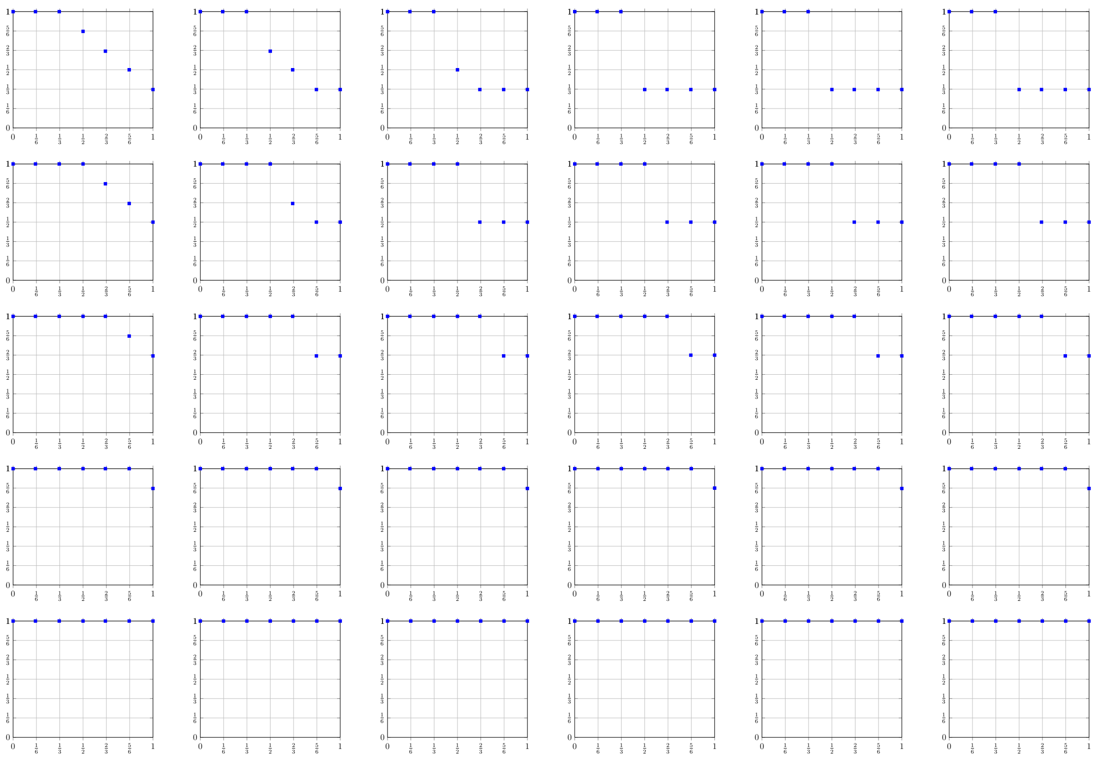


Below the six implications are shown when $z = {}^1\!/_6$ is fixed:



When $z = {}^2\!/_6$, $z = {}^3\!/_6$, $z = {}^4\!/_6$, $z = {}^5\!/_6$ and $z = 1$ are fixed, the implications are depicted in Table 8.

Table 8: Implications related to $z = {}^2\!/_6$, $z = {}^3\!/_6$, $z = {}^4\!/_6$, $z = {}^5\!/_6$ and $z = 1$ (by rows).



As a consequence, a relation between implications and degrees of preference can be provided. This is shown in Table 9.

Table 9: Relation between implications and degrees of preference given by μ_{P}

Implication	Preferences
$\nwarrow_{T_1}$	Strong preference
$\nwarrow_{T_2}$	Preference
$\nwarrow_{T_3}$	Normal
$\nwarrow_{T_4}$	Dislike
$\nwarrow_{T_6}$	Rejection

Definition 20. The family $\{\nwarrow_{T_1}, \nwarrow_{T_2}, \nwarrow_{T_3}, \nwarrow_{T_4}, \nwarrow_{T_6}\}$ of residuated implications associated with the divisible discrete t-norms T_i, with $i \in \{1, 2, 3, 4, 6\}$ will be called family of the five preference discrete implications on $[0, 1]_6$ (*5Pdi-family*, in short).

Consequently, we can define the fuzzy notion of "preference" in FCA throughout the membership function μ_{P} defined on the set of attributes/objects to a small qualitative scale {strong preference, preference, normal, dislike, rejection}, which can be identically understood by the user. Furthermore, it is also justified how it can be mapped into a qualitative scale (the 5Pdi-family).

The following section shows that the 'uniformity' in the differences, the regularity in the distribution of the steps and the number of these steps are proper features of this family.

3.5. Comparison with other families of implications

This section will compare the previous family with the dual one (Option (1)) and with the Łukasiewicz implication family, which is another interesting family that has been considered in several papers. For example, in [14] the authors tried to consider it in order to avoid the noise in the data, considering different noise levels determined by means of rough membership functions.

Following a dual procedure to the one given to obtain the t-norms in Table 5, we can obtain the t-norms $T_i^*\colon [0, 1]_6 \times [0, 1]_6 \to [0, 1]_6$, with $i \in$

$\{1, 2, 3, 4, 5, 6\}$, which have the following idempotent elements:

$$\begin{aligned}
&\mathrm{Idem}(T_1)=\{0, 1\}\\
&\mathrm{Idem}(T_2)=\{0, 1/6, 1\}\\
&\mathrm{Idem}(T_3)=\{0, 1/6, 2/6, 1\}\\
&\mathrm{Idem}(T_4)=\{0, 1/6, 2/6, 3/6, 1\}\\
&\mathrm{Idem}(T_5)=\{0, 1/6, 2/6, 3/6, 4/6, 1\}\\
&\mathrm{Idem}(T_6)=\{0, 1/6, 2/6, 3/6, 4/6, 5/6, 1\} = [0, 1]_6
\end{aligned}$$

From Table 10 we have that the number of steps between two consecutive implications is the same as in Table 7, however, they are not uniform as the Euclidean distances show.

Table 10: Manhattan distance D_1 and Euclidean distance D_2

	D_1	D_2
$\lvert\nwarrow_{T_1} - \nwarrow_{T_2}\rvert$	2.500	1.236
$\lvert\nwarrow_{T_2} - \nwarrow_{T_3}\rvert$	1.666	0.913
$\lvert\nwarrow_{T_3} - \nwarrow_{T_4}\rvert$	1.000	0.624
$\lvert\nwarrow_{T_4} - \nwarrow_{T_5}\rvert$	0.500	0.373
$\lvert\nwarrow_{T_5} - \nwarrow_{T_6}\rvert$	0.166	0.166

Therefore, this possibility provides a worse family than the proposed one since more irregular differences arise.

The other family to be considered is the one given by the discretization of the Łukasiewicz implications in Lemma 14 on the granular interval $[0, 1]_6$. Although one can think that this family provides a big range of implications, we see in Figure 1 the representation of the implications with $\alpha = 0$ and $\alpha = 1$, which indicates that a remarkable difference between them, on the granular interval $[0, 1]_6$, does not exist. This fact shows that they are equal except when $x = 0$ and $x = 1/6$, in which (small) differences are presented. Therefore, the relations to the degrees of preference will not be very striking. Specifically, the distances D_1 and D_2 between these extreme implications are 1.666 and 0.666, respectively, which are values similar to only one step in the 5Pdi-family (Table 7).

The distances D_1 and D_2 associated with the discretization of the Łukasiewicz implications on $[0, 1]_6$ have been computed and are shown in Table 11. Comparing the obtained results with the ones given in Table 7, we have

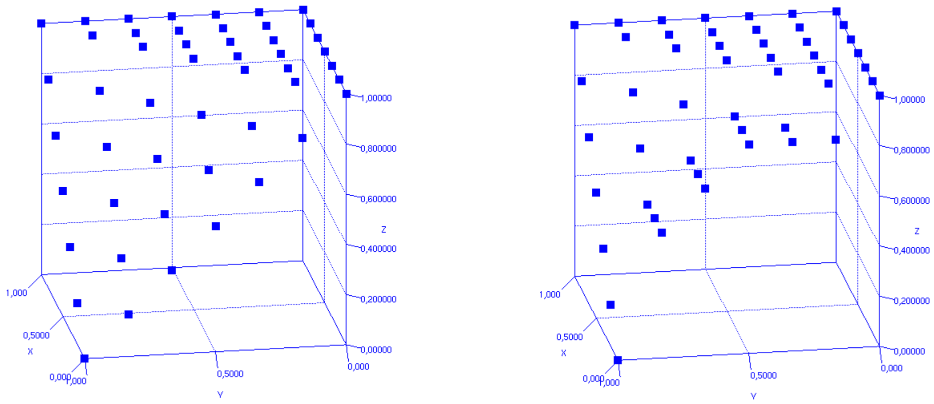


Figure 1: The discrete Łukasiewicz implications with $\alpha = 0$ (left) and $\alpha = 1$ (right).

that the differences among them are smaller. As a consequence, a correspondence between these implications and degrees of preference will not provide a sufficient impact in the results, that is, the degrees of preference will not be mapped into a suitable qualitative scale, sufficiently different to interpret the cognitive meaning of the preferences. Thus, enough different implications cannot be considered to represent different (around 7) preference degrees from the Łukasiewicz implications family on $[0, 1]_6$.

Table 11: Distances D_1 and D_2 associated with the Łukasiewicz implications on $[0, 1]_6$

	D_1	D_2
$\lvert \nwarrow_{L^{0.0}} - \nwarrow_{L^{0.2}} \rvert$	0.0	0.0
$\lvert \nwarrow_{L^{0.2}} - \nwarrow_{L^{0.4}} \rvert$	0.666	0.333
$\lvert \nwarrow_{L^{0.4}} - \nwarrow_{L^{0.6}} \rvert$	0.166	0.166
$\lvert \nwarrow_{L^{0.6}} - \nwarrow_{L^{0.8}} \rvert$	0.500	0.288
$\lvert \nwarrow_{L^{0.8}} - \nwarrow_{L^{1.0}} \rvert$	0.333	0.236

4. Worked out example

This section considers the example introduced in [37], which is related to the selection of a suitable journal to which a scientific paper can be submitted.

The following sets of attributes (parameters in the ISI Journal Citation Report) and objects (ISI journals) were taken into account:

$$A = \{\text{Impact Factor}, \text{Immediacy Index}, \text{Cited Half-Life}, \text{Best Position}\}$$
$$B = \{\text{AMC}, \text{CAMWA}, \text{FSS}, \text{IEEE-FS}, \text{IJGS}, \text{IJUFKS}\}$$

where the "best position" means the best quartile of the different categories under which the journal is included, and the journals considered are Applied Mathematics and Computation (AMC), Computer and Mathematics with Applications (CAMWA), Fuzzy Sets and Systems (FSS), IEEE transactions on Fuzzy Systems (IEEE-FS), International Journal of General Systems (IJGS), International Journal of Uncertainty Fuzziness and Knowledge-based Systems (IJUFKS). We will consider the multi-adjoint frame $([0,1]_6, \leq, T_i)$, where T_i denotes the conjunctors defined in Section 3.4, with $i \in \{1,2,3\}$. The fuzzy relation between them $R\colon A \times B \to P$ is the normalization to the unit interval $[0,1]$ of the information in the JCR, see Table 12.

Table 12: Fuzzy relation between the objects and the attributes.

R	AMC	CAMWA	FSS	IEEE-FS	IJGS	IJUFKS
Impact Factor	$2/6$	$1/6$	$4/6$	$5/6$	$3/6$	$2/6$
Immediacy Index	$1/6$	0	$2/6$	$1/6$	$1/6$	0
Cited Half-Life	$2/6$	$4/6$	1	$4/6$	$5/6$	$3/6$
Best Position	$4/6$	$3/6$	1	1	$3/6$	$2/6$

In order to choose a *suitable* journal to submit the scientific paper, we will consider the journals with a high impact factor, a medium immediacy index, a relatively big half-life and with not a bad position in the listing of the category. According to these criteria, we define the fuzzy notion of suitability in the context (A, B, R, σ) by the fuzzy subset $f\colon A \to [0,1]_6$ below:

$$f(\text{Impact Factor}) = 5/6, \qquad f(\text{Immediacy Index}) = 3/6,$$
$$f(\text{Cited Half-Life}) = 4/6, \qquad f(\text{Best Position}) = 3/6$$

We are interested in finding a multi-adjoint concept which represents the suitable journal as defined by the fuzzy set f. In order to determine this concept, we will compute $f^{\downarrow^{T_3}}$ for the t-norm T_3 defined in Section 3.4. Notice that the grades of the fuzzy sets $f^{\downarrow^{T_3}}$ are interpreted as degrees of preference, that is, $f^{\downarrow^{T_3}}(b)$ represents an intensity of preference in favor of object $b \in B$ [4, 20]. Considering the concept-forming operations given in Section 3.4, the required computation is:

$$\begin{aligned} f^{\downarrow^{T_3}}(\text{AMC}) &= \inf\{R(a, \text{AMC}) \nwarrow_{T_3} f(a) \mid a \in A\} \\ &= \inf\{{}^2\!/_6 \nwarrow_{T_3} {}^5\!/_6, {}^1\!/_6 \nwarrow_{T_3} {}^3\!/_6, {}^2\!/_6 \nwarrow_{T_3} {}^4\!/_6, {}^4\!/_6 \nwarrow_{T_3} {}^3\!/_6\} \\ &= {}^2\!/_6 \end{aligned}$$

Analogously, we carry out computations for the other journals:

$$\begin{array}{lll} f^{\downarrow^{T_3}}(\text{AMC}) = {}^2\!/_6 & f^{\downarrow^{T_3}}(\text{FSS}) = {}^3\!/_6 & f^{\downarrow^{T_3}}(\text{IJGS}) = {}^2\!/_6 \\ f^{\downarrow^{T_3}}(\text{CAMWA}) = {}^1\!/_6 & f^{\downarrow^{T_3}}(\text{IEEE-FS}) = {}^2\!/_6 & f^{\downarrow^{T_3}}(\text{IJUFKS}) = {}^1\!/_6 \end{array}$$

Note that as no preference has been considered, that is, the implication associated with the linguistic label *normal* has been assumed, that is, the membership function μ_{P} is constantly *normal*. Indeed, if we take into account another degree of preference for all the journals, we will obtain the same result as we see below.

The first column arises when μ_{P} is constantly the linguistic label *strong preference* and the second one when it is constantly *preference*. In both cases, although the values are different, the ranking is the same as it is expected.

$$\begin{array}{ll} f^{\downarrow^{T_1}}(\text{AMC}) = {}^3\!/_6 & f^{\downarrow^{T_2}}(\text{AMC}) = {}^2\!/_6 \\ f^{\downarrow^{T_1}}(\text{CAMWA}) = {}^2\!/_6 & f^{\downarrow^{T_2}}(\text{CAMWA}) = {}^1\!/_6 \\ f^{\downarrow^{T_1}}(\text{FSS}) = {}^5\!/_6 & f^{\downarrow^{T_2}}(\text{FSS}) = {}^4\!/_6 \\ f^{\downarrow^{T_1}}(\text{IEEE-FS}) = {}^4\!/_6 & f^{\downarrow^{T_2}}(\text{IEEE-FS}) = {}^3\!/_6 \\ f^{\downarrow^{T_1}}(\text{IJGS}) = {}^4\!/_6 & f^{\downarrow^{T_2}}(\text{IJGS}) = {}^3\!/_6 \\ f^{\downarrow^{T_1}}(\text{IJUFKS}) = {}^3\!/_6 & f^{\downarrow^{T_2}}(\text{IJUFKS}) = {}^2\!/_6 \end{array}$$

These results lead us to conclude the most suitable journal is FSS, when no preference is considered among the set of objects. Note that, it is the same conclusion obtained in [37].

Now, if the user prefers the journals in the Artificial Intelligence category (IEEE-FS and IJUFKS), then (s)he must choose the linguistic label *preference* for the journals in this category and *normal* for the rest of journals. Then, the membership function associated with the "preference" of the user is defined as:

$$\mu_{\mathrm{P}}(b) = \begin{cases} \textit{normal} & \text{if } b \in \{\text{AMC}, \text{CAMWA}, \text{FSS}, \text{IJGS}\} \\ \textit{preference} & \text{if } b \in \{\text{IEEE-FS}, \text{IJUFKS}\} \end{cases}$$

and, as a consequence, the system will automatically consider the context (A, B, R, σ'), where $\sigma'(b) = T_3$ for every $b \in B_1$ and $\sigma'(b) = T_2$ for every $b \in B_2$, being $B_1 = \{\text{AMC}, \text{CAMWA}, \text{FSS}, \text{IJGS}\}$ and $B_2 = \{\text{IEEE-FS}, \text{IJUFKS}\}$.

$$\begin{array}{lll} f^{\downarrow^{T_3}}(\text{AMC}) = {}^2\!/\!_6 & f^{\downarrow^{T_3}}(\text{FSS}) = {}^3\!/\!_6 & f^{\downarrow^{T_2}}(\text{IEEE-FS}) = {}^3\!/\!_6 \\ f^{\downarrow^{T_3}}(\text{CAMWA}) = {}^1\!/\!_6 & f^{\downarrow^{T_3}}(\text{IJGS}) = {}^2\!/\!_6 & f^{\downarrow^{T_2}}(\text{IJUFKS}) = {}^2\!/\!_6 \end{array}$$

Considering this particular preference degrees, the journals FSS and IEEE-FS have the same degree of suitability.

If the user has a *strong preference* with respect to the Artificial Intelligence category journal instead of a *normal* preference, the membership function μ_{P} changes and so, in this case, the system assigns the t-norm T_1 to every object in B_2 and the obtained results indicate that the best journal is IEEE-FS, according to our preferences.

$$\begin{array}{lll} f^{\downarrow^{T_3}}(\text{AMC}) = {}^2\!/\!_6 & f^{\downarrow^{T_3}}(\text{FSS}) = {}^3\!/\!_6 & f^{\downarrow^{T_1}}(\text{IEEE-FS}) = {}^4\!/\!_6 \\ f^{\downarrow^{T_3}}(\text{CAMWA}) = {}^1\!/\!_6 & f^{\downarrow^{T_3}}(\text{IJGS}) = {}^2\!/\!_6 & f^{\downarrow^{T_1}}(\text{IJUFKS}) = {}^3\!/\!_6 \end{array}$$

Just as it was shown in [37], the fact that we assign a *strong preference* to a specific subset of journal does not guarantee that the best choice belongs to that subset. In this example, if we consider the following notion of suitability:

$$\begin{array}{rl} f_1(\text{Impact Factor}) = {}^4\!/\!_6, & f_1(\text{Immediacy Index}) = {}^2\!/\!_6, \\ f_1(\text{Cited Half-Life}) = {}^3\!/\!_6, & f_1(\text{Best Position}) = {}^3\!/\!_6 \end{array}$$

we obtain the results listed below:

$$\begin{array}{lll} f^{\downarrow^{T_3}}(\text{AMC}) = {}^2\!/\!_6 & f^{\downarrow^{T_3}}(\text{FSS}) = 1 & f^{\downarrow^{T_1}}(\text{IEEE-FS}) = {}^5\!/\!_6 \\ f^{\downarrow^{T_3}}(\text{CAMWA}) = {}^1\!/\!_6 & f^{\downarrow^{T_3}}(\text{IJGS}) = {}^3\!/\!_6 & f^{\downarrow^{T_1}}(\text{IJUFKS}) = {}^4\!/\!_6 \end{array}$$

It is easy to observe that the best journal is FSS followed by IEEE-FS. Consequently, our choice will depend on our preferences (definition of the membership function μ_{P}) and also on our definition of suitable journal. This level of flexibility and enrichment holds thanks to the consideration of the multi-adjoint paradigm.

5. Conclusions and future work

This paper has studied the fuzzy notion of "preference" in FCA, offering an efficient family of implications (the 5Pdi-family), which can be considered to establish preferences among the attributes or/and the objects in the general concept lattice framework of the multi-adjoint concept lattices. Thanks to the detailed study, we have introduced a membership function μ_{P}, providing a correspondence between this family and a small qualitative scale of linguistic labels *strong preference*, *preference*, *normal*, *dislike* and *reject*, which can identically be understood by several individual and compared. The set of truth values has been bounded to seven following the magic Miller's Law, which offers that any user, with no knowledge on fuzzy operations, can use the different degrees of preference in a practical database in which the flexible multi-adjoint concept lattice can be used. Moreover, we have shown several interesting properties of the idempotent elements of a discrete divisible t-norm, which have been used to determine the considered family of implications. We will also consider UCI datasets to apply the developments presented in this paper.

Furthermore, other possible families of implications have been analyzed, such as the Łukasiewicz implication family, but they do not satisfy the minimal required properties in order to offer a good correspondence with the preference levels.

In the future, more properties and trends will be studied, such as the non-commutative case. Furthermore, the 5Pdi-family will be applied to real problems, such as the ones considered from our participation in the COST Action DigForASP (CA17124), whose main goal is the application of mathematics, artificial intelligence and automatic reasoning tools to digital forensic. In addition, the introduced structure will be exported and adapted to other frameworks, such as Fuzzy Rough Set Theory [15], Property-Oriented Concept Lattices and Object-Oriented Concept Lattices [33].

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